Symbolic representation and classification of integrable systems

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1   Introduction

Among nonlinear partial differential equations (PDEs) there is an exceptional class of so-called *integrable* equations. Integrable equations can be studied with the same completeness as linear PDEs, at least in principle. They possess a rich set of exact solutions and many hidden properties such as infinite hierarchies of symmetries, conservation laws, etc.

There are two known classes of integrable equations. One class is linearisable equations, i.e. equations related to linear ones by differential substitutions. For example, the famous Burgers equation

$$u_t = u_{xx} + 2uu_x$$

can be linearised by a differential substitution $u = \phi_x/\phi$ (the Cole-Hopf transformation). In the new variable ϕ it takes the form of a linear heat equation:

$$\phi_t = \phi_{xx}.$$

Another class is equations solvable by the inverse scattering transform method (such as the Korteweg de–Vries equation $u_t = u_{xxx} + 6uu_x$). There is massive literature on integrable equations, their solutions and properties (see for example monographs [1, 2, 3, 4, 5] and references therein).

In the *symmetry approach* the existence of higher symmetries of a PDE, or more precisely, of an infinite hierarchy of higher symmetries is regarded as the definition of its integrability. Existence of a finite number of symmetries of a partial differential equation may not secure its integrability [6, 7, 8, 9].

The symmetry approach based on a concept of formal recursion operator has been formulated and developed in works of Shabat and co-authors (see for example review papers [10, 11, 12, 13]). It has been shown that the existence of an infinite hierarchy of symmetries or local conservation laws or a possibility to linearise a certain equation imply the existence of a formal recursion operator. Formal recursion operator is carrying information about integrability and is not sensitive to lacunas in the infinite hierarchy of symmetries or conservation laws. The conditions of its existence give integrability conditions for the equation which can be formulated in a form of an infinite sequence of canonical densities and encode many hidden properties of the equation. If a density is non-trivial (i.e. is not a total derivative) it provides a density of a local conservation law of the equation. For linearisable equations all densities except a finite number are trivial. The sequence of canonical densities is invariant with respect to invertible (and almost invertible [14]) transformations.

It was the first approach enabling to give a complete description of integrable evolutionary equations of the form

$$
\begin{aligned}
u_t &= f(u_{xx}, u_x, u, x, t)\,, \\
u_t &= \partial_x^n u + f(\partial_x^{n-1} u, \ldots, u_x, u)\,, \quad n = 3, 4, 5\,,
\end{aligned}
$$

where f is a smooth function of its arguments [15, 16, 17]. Integrable differential-difference equation of the form

$$
u_{n,t} = f(u_{n-1}, u_n, u_{n+1})\,, \qquad n \in \mathbb{Z}
$$

have been classified in [18, 19]. It can also be applied to systems of equations. In particularly a complete classification of systems of two equations of the form

$$
\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}, \mathbf{u}_x)\,, \quad \det(A(\mathbf{u})) \neq 0\,, \quad \mathbf{u} = (u, v)^T \tag{1}
$$

possessing infinite hierarchies of local conservation laws has been given in [20, 21, 14]. Existence of higher conservation laws immediately implies that the trace of $A(\mathbf{u})$ vanishes. System (1) can still possess an infinite hierarchy of symmetries even if the trace of $A(\mathbf{u})$ is not equal to zero. In the latter case the system is linearisable by a Cole-Hopf type differential substitution. In the case of $A(\mathbf{u})$ being a unit matrix integrable systems of the type (1) have been listed in [22]. A complete classification of polynomial homogeneous

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integrable system of the form (1) in the case when matrix $A(\mathbf{u})$ is constant and has two distinct eigenvalues is given in [23].

In this approach it is not assumed that equations are polynomial or rational functions of independent variables and their derivatives. A disadvantage is that for every fixed order of the equation the integrability conditions have to be derived from scratch and thus it is difficult to draw a global picture, i.e., in all orders. Also, this approach is heavily based on the concept of locality which makes it difficult to apply to integro-differential, non-evolutionary and multi-dimensional systems.

In this article we would like to give a brief account of recent development of the symmetry approach. The progress has been achieved mainly due to a symbolic representation of the ring of differential polynomials which enable us to use powerful results from algebraic geometry and number theory. Symbolic representation (an abbreviated form of the Fourier transformation) has been originally applied to the theory of integrable equations by Gel'fand and Dikii [24]. Symmetry approach in symbolic representation has been formulated and developed to the problem of the global classification of integrable evolutionary equations in [25, 26, 27]. In symbolic representation the existence of infinite hierarchy of symmetries is linked with factorisation properties of an infinite sequence of multi-variable polynomials. Symbolic representation is a suitable tool to study integrability of noncommutative [28], non-evolutionary [29, 30, 31], non-local (integro–differential) [32], multi–component [23, 33, 34] and multi–dimensional equations [35]. It is convenient for testing integrability of a given system, provides useful information on the structure of the symmetry hierarchy and suitable for global classification of integrable equations. In this framework it is natural to define approximate symmetries and approximate integrability. Study and classification of approximately integrable equations is a new and unexplored area of research with a considerable potential for applications. Symmetry approach in symbolic representation has certain drawbacks due to a restriction to the ring of differential polynomials, which can be amended in some cases by suitable extensions of the ring.

In literature one can find other attempts to describe integrable systems based on properties of solutions, such as the Painlevé property in the analytical theory (see for example [2, 3, 4]), existence of three soliton solutions for bi-linear (Hirota) representation (see the article by Hietarinta in this book), elasticity of soliton collisions in numerical experiments, etc. Existence of one nontrivial symmetry of an prior fixed order can be used for isolation of integrable equations [36, 37]. Every method has certain advantages and disadvantages. The reader can judge the power of each approach by the results obtained, their completeness and generality.

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Our article is organised as following. In the next Section 2 we give basic definitions and notations. We define symmetries, approximate symmetries and formal recursion operator. Then we introduce symbolic representation of the ring of differential polynomials. A generalisation to several dependent variables is given in the last part of this section. In Section 3, using approximate symmetries in symbolic representation, we study the structure of the Lie algebra of symmetries. In symbolic representation the existence of approximate symmetries can be reformulated in terms of factorisation properties of polynomials. Existence of one nontrivial symmetry enables us to constructively extend any approximate symmetry of degree 3 to any degree (Theorem 4). In symbolic representation conditions for the existence of formal recursion operator lead to a simple test for integrability. Here a state of art result is a global classification of integrable homogeneous evolutionary equations (Section 3.2). The result of the classification can be accounted as following: *Integrable equations are symmetries (members of infinite hierarchies) of nonlinear PDEs of orders 2, 3 or 5. Thus it is sufficient to classify integrable equations of order 2, 3 or 5. There is only a finite number of such equations (namely 10) and the corresponding hierarchies of symmetries* (Theorem 9). In Section 4 we apply our method to non-local (Benjamin-Ono and Camassa-Holm type) equations. It requires a non-local extension of the ring of differential polynomials and symbolic representation proved to be a suitable language to tackle the problem. In Section 5 we present results on classification of Boussinesq type equations. In the case of even order equations we use conditions following from the existence of formal recursion operator. Together with classification results for orders 4 and 6 we present three new integrable equations of order 10. A global classification of all integrable odd order Boussinesq type equation is given in section 5.2. A generalisation of the symbolic approach to $2 + 1$–dimensional equations enables us (Section 6) to study the structure of symmetries of Lax integrable equations and to prove the conjecture on the structure of non-local terms. Finally we discuss a progress in the problem of classification of systems of integrable equations. In particularly we have found two integrable systems of order 5 which we believe are new.

2 Symmetries and formal recursion operators in symbolic representation

2.1 Differential polynomials

We shall adopt the following notations: u_n denotes n-th derivative $\partial_x^n u$ of the dependent variable u. In particular, u_0 denote the function u itself (often we shall omit the zero index of u_0 and simply write u).

A u-monomial is a finite product of the form

$$u_0^{\alpha_0} u_1^{\alpha_1} \cdots u_k^{\alpha_k},$$

where all exponents $\alpha_0, \ldots, \alpha_k$ are non-negative integers ($\alpha_s \in \mathbb{Z}_{\geq 0}$), and the total degree is $|\alpha| = \alpha_0 + \alpha_1 + \cdots \alpha_k > 0$.

A finite sequence $\alpha = (\alpha_0, \ldots, \alpha_k)$ can be seen as an element of a semi-group $\mathbb{Z}_{\geq 0}^{\infty *}$ of infinite sequences of non-negative integers, such that only a finite number of entries in a sequence are non-zero and there is at least one nonzero entry. There is an obvious bijection between the set of all u-monomials and $\mathbb{Z}_{\geq 0}^{\infty *}$. We can simplify the notations as follows:

$$u^\alpha = u_0^{\alpha_1} u_1^{\alpha_2} \cdots u_k^{\alpha_k}.$$

A *differential polynomial* f in variables $u_0, u_1, \ldots$ with coefficients in $\mathbb{C}$ is a finite linear combination of u-monomials, i.e.,

$$f = \sum_{\alpha \in A} a_\alpha u^\alpha, \qquad a_\alpha \in \mathbb{C},$$

where the sum is over a finite set $A = \{\alpha \, | \, \alpha \in \mathbb{Z}_{\geq 0}^{\infty *}\}$. The set of all such differential polynomials is denoted $\mathcal{R}$. It is a ring with usual addition and multiplication of polynomials. Moreover, it is a differential ring and $\mathbb{C} \notin \mathcal{R}$. The linear operator

$$D_x = \sum_{k \geq 0} \left(u_{k+1} \frac{\partial}{\partial u_k} \right) \tag{2}$$

is a derivation of $\mathcal{R}$ corresponding to the total x-derivative. Operator D_x acting on an element $f \in \mathcal{R}$ results in a finite sum depending on the choice of f. Therefore, we do not indicate the upper limit for the summation in the definition of D_x and all other operators defined in this article.

It is easy to verify that monomials u^α are eigenvectors of the following commuting linear operators

$$D_u = \sum_{k \geq 0} u_k \frac{\partial}{\partial u_k}, \qquad X_u = \sum_{k \geq 1} \left(k u_k \frac{\partial}{\partial u_k} \right) \tag{3}$$

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with $D_u(u^\alpha) = |\alpha|u^\alpha$, and $X_u(u^\alpha) = d_\alpha u^\alpha$, where $d_\alpha = \sum_{k\geq 1}(k\alpha_k)$.

Thus the ring $\mathcal{R}$ is graded and is a direct sum of eigenspaces

$$\mathcal{R} = \bigoplus_{n\in\mathbb{N}} \mathcal{R}^n = \bigoplus_{n,p\in\mathbb{N}} \mathcal{R}^n_{p-1},$$

$$\mathcal{R}^n = \{f \in \mathcal{R} \mid D_u(f) = nf\}, \quad \mathcal{R}^n_p = \{f \in \mathcal{R}^n \mid X_u(f) = pf\}.$$

If $f \in \mathcal{R}^n_p$, $g \in \mathcal{R}^m_q$, then $f \cdot g \in \mathcal{R}^{n+m}_{p+q}$. Simply speaking, $\mathcal{R}^n_p$ is a linear subspace of homogeneous differential polynomials such that each monomial has: (i). the number of u and its derivatives being n; (ii). the total number of derivatives being p. For example,

$$u_1^2 u_7 + 2u_2 u_3 u_4 - u_0^2 u_9 \in \mathcal{R}^3_9 \subset \mathcal{R}^3.$$

In some applications it is convenient to introduce weighted homogeneous polynomials. Let us assume that dependent variable u has a weight λ which is a fixed rational number. We define a linear differential operator

$$W_\lambda = \lambda D_u + X_u, \quad W_\lambda : \mathcal{R} \to \mathcal{R}.$$

Differential monomials are eigenvectors of W_λ and the spectrum of W_λ is a set $S_\lambda = \{n\lambda + m - 1 \mid n, m \in \mathbb{N}\}$. We can decompose $\mathcal{R}$ in a direct sum of eigenspaces

$$\mathcal{R} = \bigoplus_{\mu \in S_\lambda} \mathcal{W}_\mu, \qquad \mathcal{W}_\mu = \{f \in \mathcal{R} \mid W_\lambda(f) = \mu f\}.$$

Elements of $\mathcal{W}_\mu$ are called λ-homogeneous differential polynomials of weight μ. For example, $u_3 + 6uu_1$ is a 2-homogeneous differential polynomial of weight 5. We have $\mathcal{W}_\mu \mathcal{W}_\nu \subset \mathcal{W}_{\mu+\nu}$. Moreover, if $\lambda > 0$ then subspaces $\mathcal{W}_\mu$ are finite dimensional.

It is useful to define the little "*oh*" order symbol.

Definition 1 *Let $f \in \mathcal{R}$. We say that $f = o(\mathcal{R}^n)$ if $f \in \bigoplus_{k>n} \mathcal{R}^k$.*

For example, $f = o(\mathcal{R}^3)$ means that the differential polynomial f does not have linear, quadratic and cubic terms in u and its derivatives.

For any two elements $f, g \in \mathcal{R}$ we define a Lie bracket

$$[f, \; g] = f_*(g) - g_*(f), \tag{4}$$

where the *Fréchet derivative* for any element $h \in \mathcal{R}$ is defined as a linear differential operator of the form

$$h_* = \sum_{k\geq 0} \frac{\partial h}{\partial u_k} D_x^k. \tag{5}$$

We say that an element $h \in \mathcal{R}$ has *order n* if the corresponding differential operator h_* is of order n.

Thus $\mathcal{R}$, treated as a linear space over $\mathbb{C}$, together with the Lie bracket (4) is an infinite dimensional Lie algebra of differential polynomials over $\mathbb{C}$. The bilinearity and skew-symmetry of the bracket (4) are obvious. The Jacobi identity can be easily verified.

The grading of $\mathcal{R}$ induces the grading of the Lie algebra of differential polynomials since we have

$$[\mathcal{R}_p^n, \mathcal{R}_q^m] \subset \mathcal{R}_{p+q}^{n+m-1} \,. \tag{6}$$

2.2 Symmetries, approximate symmetries and formal recursion operator

In this section, for the sake of simplicity, we give definitions suitable for evolutionary equations

$$u_t = F, \qquad F \in \mathcal{R} \,. \tag{7}$$

These definitions will be later extended to non-evolutionary equations and the multi-component systems of evolutionary equations.

Evolutionary equation (7) defines a derivation $D_F : \mathcal{R} \mapsto \mathcal{R}$:

$$D_F(a) = a_*(F).$$

In this notation the derivative $D_x = D_{u_1}$ and it is in agreement with (3). Sometimes, for simplification of notations, we will denote D_F as D_t.

Definition 2 *A differential polynomial $G \in \mathcal{R}$ is said to be a symmetry (a generator of an infinitesimal symmetry) for an evolutionary partial differential equation (7) if the Lie bracket of F and G vanishes, i.e., $[F, \, G] = 0$.*

If G is a symmetry, then evolutionary equation $u_\tau = G$ is compatible with (7). There are many other equivalent definitions of symmetry (see for example [10, 12]). Elements of $\mathcal{R}$ do not depend on x, t explicitly. Thus our definition does not include space and time dependent symmetries such as dilatation and Galilean symmetries. In this article, when we are talking about symmetries we do mean space and time independent symmetries.

Symmetries form a subalgebra $C_{\mathcal{R}}(F) = \{G \in \mathcal{R} \,|\, [F, \, G] = 0\}$ which is the centraliser of F (it immediately follows from the Jacobi identity). Since $F \in \mathcal{R}$ does not depend on x, t explicitly, then any equation (7) possesses *trivial* symmetries $u_1, F \in C_{\mathcal{R}}(F)$ corresponding to translations in space and time.

Definition 3 *Equation (7) is said to be integrable if its algebra of symmetries $C_{\mathcal{R}}(F)$ is infinite dimensional.*

For nonlinear equations of the form

$$u_t = u_n + f(u_{n-1}, \ldots, u) \qquad n \geq 2, \tag{8}$$

it is easy to show that the algebra of symmetries $C_{\mathcal{R}}(u_t)$ is commutative. Moreover, the symmetries $G \in \mathcal{R}$ must have a linear term (see section 3.1).

Having in mind the gradation (6) of the Lie algebra of differential polynomials we represent the right hand side of equation (7) and infinitesimal generators of symmetries in the form

$$F = f_1 + f_2 + \cdots, \quad G = g_1 + g_2 + \cdots, \qquad f_k, g_k \in \mathcal{R}^k$$

and study them in the sequence of terms, i.e., linear, quadratic, cubic, etc.

Definition 4 *A differential polynomial $G \in \mathcal{R}$ is said to be an approximate symmetry of degree p for evolutionary partial differential equation (7) if $[F, \; G] = o(\mathcal{R}^p)$.*

Equation (8) possesses infinitely many approximate symmetries of degree 1. An equation may possess approximate symmetries of degree 2, but fail to possess approximate symmetries of degree 3. An integrable equation possesses infinitely many approximate symmetries of any degree. In the next section, using symbolic representation we formulate the necessary and sufficient conditions for the existence of approximate symmetries of arbitrary degree.

For example, equation

$$u_t = u_5 + 5uu_1, \tag{9}$$

has approximate symmetry of degree 2 with a generator

$$G = u_7 + 7uu_3 + 14u_1u_2.$$

Indeed,

$$[G, u_5 + 5uu_1] = 210u_1^2u_2 + 105uu_2^2 + 105uu_1u_3 = o(\mathcal{R}^2).$$

Moreover, equation (9) has infinitely many approximate symmetries of degree 2 (this fact will become obvious in the next section), but fails to have approximate symmetries of degree 3 and thus it is not integrable.

It follows from the Jacobi identity and (6) that approximate symmetries of degree n form a subalgebra of $\mathcal{R}$ which we denote $C_{\mathcal{R}}^n(F)$. Obviously

$$\mathcal{R} = C_{\mathcal{R}}^1(F) \supset C_{\mathcal{R}}^2(F) \supset C_{\mathcal{R}}^3(F) \supset \cdots \supset C_{\mathcal{R}}^\infty(F) = C_{\mathcal{R}}(F) \,.$$

Formal pseudo-differential series, which for simplicity we shall call formal series, are defined as

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots \quad a_k \in \mathcal{R} \,. \tag{10}$$

The product of two formal series is defined by

$$aD_x^k \circ bD_x^m = a(bD_x^{m+k} + C_k^1 b_1 D_x^{k+m-1} + C_k^2 b_2 D_x^{k+m-2} + \cdots), \tag{11}$$

where $b_j = D_x^j(b)$, $k, m \in \mathbb{Z}$ and the binomial coefficients are defined as

$$C_n^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} \,.$$

This product is associative.

Definition 5 *A formal series*

$$\Lambda = l_m D_x^m + \cdots + l_0 + l_{-1} D_x^{-1} + \cdots \,, \quad l_k \in \mathcal{R} \tag{12}$$

is called a formal recursion operator for equation (7) if

$$D_F(\Lambda) = F_* \circ \Lambda - \Lambda \circ F_* \,. \tag{13}$$

In literature a formal recursion operator is also called a formal symmetry of equation (7).

The central result of the Symmetry Approach can be represented by the following Theorem, which we attribute to Shabat:

Theorem 1 *If equation (7) has an infinite hierarchy of symmetries of arbitrary high order, then a formal recursion operator exists and its coefficients can be found recursively.*

The Theorem states that for integrable equations, i.e. equations possessing an infinite hierarchy of higher symmetries, one can solve equation (13) and determine recursively the coefficients $l_m, l_{m-1}, \ldots$ of Λ such that all these coefficients will belong to the ring $\mathcal{R}$. The solvability conditions of equation (13) can be formulated in an elegant form of a canonical sequence of local conservation laws of equation (7). They provide powerful necessary conditions of integrability. These conditions can be used for testing for integrability for a given equation or even for a complete description of integrable equations of a particular order. A detailed description of the Symmetry Approach including the proof of the above theorem and classification results for integrable PDEs and systems of PDEs based on the concept of formal recursion operator can be found in review papers [10, 11, 12, 13].

2.3 Symbolic representation

Symbolic representation transforms the problems in differential algebra into the ones in algebra of symmetric polynomials. This enables us to use powerful results from Diophantine equations, algebraic geometry and commutative algebra. Symbolic representation is widely used in theory of pseudo-differential operators. To integrable systems it was first applied by Gel'fand and Dickey [24] and further developed in works of Beukers, Sanders and Wang [7, 25, 26]. Symbolic representation can be viewed as a simplified notation for a Fourier transform [29].

In order to define the symbolic representation $\hat{\mathcal{R}} = \oplus \hat{\mathcal{R}}^n$ of the ring (and Lie algebra) of differential polynomials $\mathcal{R} = \oplus \mathcal{R}^n$, we first define an isomorphism of the linear spaces $\mathcal{R}^n$ and $\hat{\mathcal{R}}^n$ and then extend it to isomorphisms of the differential ring and Lie algebra equipping $\hat{\mathcal{R}}$ with the multiplication, derivation and Lie bracket.

Symbolic transform defines a linear isomorphism between the space $\mathcal{R}^n$ of differential polynomials of degree n and the space $\mathbb{C}[\xi_1, \ldots, \xi_n]^{\mathcal{S}_n^\xi}$ of algebraic symmetric polynomials in n variables, where $\mathcal{S}_n^\xi$ is a permutation group of n variables $\xi_1, \ldots, \xi_n$. Elements of $\hat{\mathcal{R}}^n$ are denoted by $\hat{u}^n a(\xi_1, \ldots, \xi_n)$, where $a(\xi_1, \ldots, \xi_n) \in \mathbb{C}[\xi_1, \ldots, \xi_n]^{\mathcal{S}_n^\xi}$. The isomorphism of linear spaces $\mathcal{R}^n$ and $\hat{\mathcal{R}}^n$ is uniquely defined by its action on monomials.

Definition 6 *The* symbolic form *of a differential monomial is defined as*

$$u_{i_1} u_{i_2} \cdots u_{i_n} \in \mathcal{R}^n \quad \longmapsto \quad \hat{u}^n \langle \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n} \rangle_{\mathcal{S}_n^\xi} \in \hat{\mathcal{R}}^n$$

where $\langle \cdot \rangle_{\mathcal{S}_n^\xi}$ denotes the average over the group $\mathcal{S}_n^\xi$ of permutation of n elements $\xi_1, \ldots, \xi_n$:

$$\langle a(\xi_1, \cdots, \xi_n) \rangle_{\mathcal{S}_n^\xi} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n^\xi} a(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}).$$

For example,

$$u_k \longmapsto \hat{u} \xi_1^k, \ u^n \longmapsto \hat{u}^n, \ uu_1 \longmapsto \frac{\hat{u}^2}{2}(\xi_1 + \xi_2),$$

$$uu_1^2 \longmapsto \frac{\hat{u}^3}{3}(\xi_2 \xi_3 + \xi_1 \xi_2 + \xi_1 \xi_3), \ u_k^3 \longmapsto \hat{u}^3 \xi_1^k \xi_2^k \xi_3^k.$$

It is easy to see that in $\hat{\mathcal{R}}$ the linear operators D_u and X_u, cf. (3) are represented by

$$\hat{D}_u = \hat{u} \frac{\partial}{\partial \hat{u}}, \qquad \hat{X}_u = \sum_{i=1} \xi_i \frac{\partial}{\partial \xi_i}.$$

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With this isomorphism the linear spaces $\hat{\mathcal{R}}_p^n$ corresponding to $\mathcal{R}_p^n$ have the property that the coefficient functions $a(\xi_1,\ldots,\xi_n)$ of symbols

$$\hat{u}^n a(\xi_1,\ldots,\xi_n) \in \hat{\mathcal{R}}_p^n$$

are homogeneous symmetric polynomials of degree p.

One of the advantages of the symbolic representation is that the action of the operator D_x, cf. (2) is very simple. Indeed, let $f \in \mathcal{R}^n$ and $f \longmapsto \hat{u}^n a(\xi_1,\ldots,\xi_n)$ then

$$D_x(f) \longmapsto \hat{u}^n a(\xi_1,\ldots,\xi_n)(\xi_1 + \cdots + \xi_n),$$

and thus $D_x^k(f) \longmapsto \hat{u}^n a(\xi_1,\ldots,\xi_n)(\xi_1 + \cdots + \xi_n)^k$.

Let $f \in \mathcal{R}^n$, $f \mapsto \hat{u}^n a(\xi_1,\ldots,\xi_n)$ and $g \in \mathcal{R}^m$, $g \mapsto \hat{u}^m b(\xi_1,\ldots,\xi_m)$, then

(i). The product $f \cdot g$ has the following symbolic representation:

$$\begin{aligned} f \cdot g \quad &\longmapsto \quad \hat{u}^n a(\xi_1,\ldots,\xi_n) \circ \hat{u}^m b(\xi_1,\ldots,\xi_m) \\ &= \hat{u}^{n+m} \langle a(\xi_1,\ldots,\xi_n) b(\xi_{n+1},\ldots,\xi_{n+m}) \rangle_{\mathcal{S}_{n+m}^\xi}. \end{aligned}$$

This defines the corresponding multiplication $\circ$ in $\hat{\mathcal{R}}$. Representation of differential monomials (Definition 6) can be deduced from $u_k \longmapsto \hat{u} \xi_1^k$ and this multiplication rule.

(ii). The Lie bracket $[f,g]$, cf. (4) is represented by

$$\begin{aligned} [f,\ g] \quad &\longmapsto \quad \hat{u}^{n+m-1} \qquad\qquad\qquad\qquad\qquad (14)\\ &\langle na(\xi_1,\ldots,\xi_{n-1},\xi_n + \cdots + \xi_{n+m-1}) b(\xi_n,\ldots,\xi_{n+m-1}) - \\ &mb(\xi_1,\ldots,\xi_{m-1},\xi_m + \cdots + \xi_{n+m-1}) a(\xi_m,\ldots,\xi_{n+m-1}) \rangle_{\mathcal{S}_{n+m-1}^\xi} \end{aligned}$$

For example, if $f \in \mathcal{R}^1$, $f \mapsto \hat{u}\omega(\xi_1)$ and $g \in \mathcal{R}^n$, $g \mapsto \hat{u}^n a(\xi_1,\ldots,\xi_n)$, then

$$[f,g] \longmapsto (\omega(\xi_1 + \cdots + \xi_n) - \omega(\xi_1) - \cdots - \omega(\xi_n))\ \hat{u}^n a(\xi_1,\ldots,\xi_n).$$

In particular, for $f = u_1$ we have $\omega(\xi_1) = \xi_1$ and $[u_1, g] = 0$ for any $g \in \mathcal{R}$. Thus u_1 is a symmetry for any evolutionary equation $u_t = g$.

Symbolic representation of differential operators (such as the Fréchet derivative (5) and formal series (10)) is motivated by the theory of linear pseudo-differential operators in Fourier representation. To operator D_x (2) we shall assign a special symbol η and the following rules of action on symbols:

$$\eta(\hat{u}^n a(\xi_1,\ldots,\xi_n)) = \hat{u}^n a(\xi_1,\ldots,\xi_n) \sum_{j=1}^n \xi_j$$

and the composition rule

$$\eta \circ \hat{u}^n a(\xi_1, \ldots, \xi_n) = \hat{u}^n a(\xi_1, \ldots, \xi_n)(\sum_{j=1}^{n} \xi_j + \eta).$$

The latter corresponds to the Leibnitz rule $D_x \circ f = D_x(f) + fD_x$. Now it can be shown that the composition rule (11) can be represented as following. Let we have two operators fD_x^q and gD_x^s such that f and g have symbols $\hat{u}^i a(\xi_1, \ldots, \xi_i)$ and $\hat{u}^j b(\xi_1, \ldots, \xi_j)$ respectively. Then

$$fD_x^q \longmapsto \hat{u}^i a(\xi_1, \ldots, \xi_i)\eta^q, \qquad gD_x^s \longmapsto \hat{u}^j b(\xi_1, \ldots, \xi_j)\eta^s$$

and

$$fD_x^q \circ gD_x^s \longmapsto \hat{u}^{i+j} \langle a(\xi_1, \ldots, \xi_i)(\eta + \sum_{m=i+1}^{i+j} \xi_m)^q b(\xi_{i+1}, \ldots, \xi_{i+j})\eta^s \rangle_{\mathcal{S}_{i+j}^\xi}. \quad (15)$$

Here the symmetrisation is taken over the group of permutation of all $i + j$ arguments $\xi_1, \ldots \xi_{i+j}$, the symbol η is not included in this set. In particularly, it follows from (15) that $D_x^q \circ D_x^s \mapsto \eta^{q+s}$. The composition rule (15) is valid for both positive and negative exponents. In the case of positive exponents it is a polynomial in η and the result is a Fourier image of a differential operator. In the case of negative exponents one can expand the result on η at $\eta \to \infty$ in order to identify it with (11). In the symbolic representation instead of formal series (10) it is natural to consider a more general object, namely formal series of the form

$$B = b(\eta) + \hat{u}b_1(\xi_1, \eta) + \hat{u}^2 b_2(\xi_1, \xi_2, \eta) + \hat{u}^3 b_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots,$$

where coefficients $b(\eta) \neq 0, b_k(\xi_1, \ldots, \xi_k, \eta)$ are rational functions of its arguments (with certain restrictions which will be discussed in the next section).

The symbolic representation of the Fréchet derivative of the element $f \longmapsto \hat{u}^n a(\xi_1, \ldots, \xi_n)$ is

$$f_* \longmapsto n\hat{u}^{n-1} a(\xi_1, \ldots, \xi_{n-1}, \eta).$$

For example, let $F = u_3 + 6uu_1$, then $F \mapsto \hat{u}\xi_1^3 + 3\hat{u}^2(\xi_1 + \xi_2)$ and

$$F_* \mapsto \eta^3 + 6\hat{u}(\xi_1 + \eta).$$

It is interesting to notice that the symbol of the Fréchet derivative is always symmetric with respect to all permutations of arguments, including the argument η. Moreover, the following obvious but useful Proposition holds [29]:

Proposition 1 *A differential operator is a Fréchet derivative of an element of $\mathcal{R}$ if and only if its symbol is invariant with respect to all permutations of its argument, including the argument η.*

The symbolic representation has been extended and proved to be very useful in the case of noncommutative differential rings [28]. In the next sections symbolic representation will be extended to the cases of many dependent variables, suitable for study of system of equations and further to the cases of non-local and multidimensional equations.

2.4 Generalisation to several dependent variables

The definitions and most of the statements formulated in the previous sections 2.1–3.1 can be easily extended to several dependents, i.e., to systems of equations. In this section we will give a brief account of definitions and some results concerning two dependent variables. A generalisation for N dependent variables is straightforward. For details see [11, 12, 13, 30, 31, 33].

Similarly to u–monomials (Section 2.1), we define v-monomials

$$v^\beta = v_0^{\beta_0} v_1^{\beta_1} \cdots v_s^{\beta_s}, \quad \beta \in \mathbb{Z}_{\geq 0}^\infty.$$

A differential polynomial f in variables $u_0, v_0, u_1, v_1, \ldots$ is a finite linear combination of the form

$$f = \sum_{(\alpha,\beta)\in A} a_{\alpha,\beta} u^\alpha v^\beta, \qquad a_{\alpha,\beta} \in \mathbb{C},$$

where the sum is taken over a finite set

$$A = \{(\alpha,\beta) \,|\, \alpha, \beta \in \mathbb{Z}_{\geq 0}^\infty, \ |\alpha| + |\beta| > 0\}.$$

It is a differential ring. We again denote it $\mathcal{R}$. Derivation D_x, (cf. (2)) is now replaced by

$$D_x = \sum_{k\geq 0} \left(u_{k+1} \frac{\partial}{\partial u_k} + v_{k+1} \frac{\partial}{\partial v_k} \right).$$

For any $f \in \mathcal{R}$ the Fréchet derivative f_* is a (row) vector operator

$$f_* = (f_{*u}, f_{*v}) = \left(\sum_{k\geq 0} \frac{\partial f}{\partial u_k} D_x^k, \quad \sum_{k\geq 0} \frac{\partial f}{\partial v_k} D_x^k \right). \tag{16}$$

Systems of two evolutionary equations we will write in vector form

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n), \tag{17}$$

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where $\mathbf{u}_k = (u_k, v_k)^T$ and $\mathbf{F} = (F_1, F_2)^T$ are vector-columns where $F_1, F_2 \in \mathcal{R}$ (the upper index T stands for the transposition).

Let us introduce an infinite dimensional linear space over $\mathbb{C}$

$$\mathcal{L} = \{(H_1, H_2)^T \mid H_1, H_2 \in \mathcal{R}\}.$$

We equip $\mathcal{L}$ with a Lie bracket

$$[\mathbf{F}, \mathbf{G}] = \mathbf{F}_*(\mathbf{G}) - \mathbf{G}_*(\mathbf{F}) \in \mathcal{L},$$

where the Fréchet derivative $\mathbf{H}_*$ for any $\mathbf{H} \in \mathcal{L}$ is defined as

$$\mathbf{H}_* = \begin{pmatrix} H_{1*u} & H_{1*v} \\ H_{2*u} & H_{2*v} \end{pmatrix}.$$

Thus $\mathcal{L}$ has a structure of an infinite dimensional Lie algebra over $\mathbb{C}$. Subalgebra of symmetries of equation (17) is the centraliser $C_{\mathcal{L}}(\mathbf{F})$ of $\mathbf{F}$ in $\mathcal{L}$ (cf. section 2.1).

Evolutionary system (17) defines a derivation $D_{\mathbf{F}} : \mathcal{R} \mapsto \mathcal{R}$

$$D_{\mathbf{F}}(a) = a_{*u}(F_1) + a_{*v}(F_2),$$

which is also a derivation of the Lie algebra $\mathcal{L}$. This derivation we often denote as D_t. In this notations derivation D_x coincides with $D_{\mathbf{u_1}}$ and $D_u + D_v$ with $D_{\mathbf{u}}$.

The ring $\mathcal{R}$ has several gradings. Here we define a *monomial degree* grading

$$\mathcal{R} = \bigoplus_{k \in \mathbb{N}} \mathcal{R}^k, \quad \mathcal{R}^k = \{a \in \mathcal{R} \mid D_{\mathbf{u}}a = ka\}.$$

We say that $a = o(\mathcal{R}^n)$ if $a \in \bigoplus_{k>n} \mathcal{R}^k$.

Lie algebra $\mathcal{L}$ inherits the gradings of $\mathcal{R}$. A *monomial degree* grading

$$\mathcal{L} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{L}^k, \quad \mathcal{L}^k = \{\mathbf{H} \in \mathcal{L} \mid D_{\mathbf{u}}\mathbf{H} = (k+1)\mathbf{H}\},$$

and thus $[\mathcal{L}^p, \mathcal{L}^q] \subset \mathcal{L}^{p+q}$ is convenient for the definition of approximate symmetries. For $\mathbf{H} \in \mathcal{L}$ we say that $\mathbf{H} = o(\mathcal{L}^n)$ if $\mathbf{H} \in \bigoplus_{k>n} \mathcal{L}^k$. Approximate symmetries of equation (17) of degree n are defined as elements of the approximate centraliser

$$C_{\mathcal{L}}^n(\mathbf{F}) = \{\mathbf{G} \in \mathcal{L} \mid [\mathbf{F}, \mathbf{G}] = o(\mathcal{L}^{n-1})\},$$

which is a subalgebra of $\mathcal{L}$.

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The *weighted gradation* is useful for the study of homogeneous systems. We assign weights $\mathbf{w} = (w_u, w_v)$ with rational entries to the vector variable $\mathbf{u}$ and define a linear operator

$$W = (w_u D_u + w_v D_v + X_u + X_v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} w_u & 0 \\ 0 & w_v \end{pmatrix}$$

with spectrum $S_W = \{(p-1)w_u + (q-1)w_v + r \,|\, p, q, r \in \mathbb{Z}_{\geq 0},\ p+q > 0\}$. Then the linear subspaces $\mathcal{L}_\mu$ in the decomposition

$$\mathcal{L} = \bigoplus_{\mu \in S_W} \mathcal{L}_\mu, \quad \mathcal{L}_\mu = \{\mathbf{H} \in \mathcal{L} \,|\, W\mathbf{H} = \mu\mathbf{H}\}$$

satisfy to

$$[\mathcal{L}_\mu, \mathcal{L}_\nu] \subset \mathcal{L}_{\mu+\nu}. \tag{18}$$

Elements of $\mathcal{L}_\mu$ we call $\mathbf{w}$–homogeneous differential polynomial vectors of weight μ.

For example, if the weight vector of variables (u, v) is $\mathbf{w} = (1/2, 1)$, then

$$\mathbf{F} = \begin{pmatrix} v_1 \\ u_2 + 3uv_1 + vu_1 - 3u^2 u_1 \end{pmatrix}$$

is $\mathbf{w}$–homogeneous element of weight $3/2$, indeed $\mathbf{F} \in \mathcal{L}_{3/2}$.

If $\mathbf{F} \in \mathcal{L}_\mu$ is a homogeneous vector, then $D_\mathbf{F}$ is a homogeneous derivation of weight μ:

$$D_\mathbf{F} \mathcal{L}_\nu \subset \mathcal{L}_{\nu+\mu}, \qquad \mathbf{F} \in \mathcal{L}_\mu.$$

From (18) it immediately follows

Proposition 2 *Let* $\mathbf{G} = \mathbf{G}_{\nu_1} + \cdots + \mathbf{G}_{\nu_m}$, $\mathbf{G}_\gamma \in \mathcal{L}_\gamma$ *be a generator of a symmetry of a homogeneous equation, then each* $\mathbf{w}$*–homogeneous component* $\mathbf{G}_{\nu_k}$ *is a generator of a symmetry.*

For evolutionary system (17) a recursion operator (a formal recursion operator) Λ can be defined as a differential or pseudo-differential operator (or a formal series)

$$\Lambda = \Lambda_k D_x^k + \Lambda_{k-1} D_x^{k-1} + \cdots, \quad \Lambda_s \in \mathrm{Mat}_{2\times 2}(\mathcal{R}),$$

which satisfies the following operator equation

$$D_\mathbf{F}(\Lambda) = \mathbf{F}_* \circ \Lambda - \Lambda \circ \mathbf{F}_* \tag{19}$$

(compare with Definition 5). If action of Λ is well defined on a symmetry $\mathbf{G}_1$, i.e. $\mathbf{G}_2 = \Lambda(\mathbf{G}_1) \in \mathcal{L}$, then $\mathbf{G}_2$ is a new symmetry of the evolutionary

system (17). Starting from a "seed" symmetry $\mathbf{G}_1$, one can build up an infinite hierarchy of symmetries $\mathbf{G}_{n+1} = \Lambda^n(\mathbf{G}_1)$, provided that each action of Λ produces an element of $\mathcal{L}$.

Symbolic representation of the ring $\mathcal{R}$ generated by two independent variables u, v is quite straightforward. It is a $\mathbb{C}$–linear isomorphism which is sufficient to define for the monomials. Suppose we have a monomial $u^\alpha v^\beta$. Let the symbolic representation for the monomial u^α be $\hat{u}^{|\alpha|} a(\xi_1, \ldots, \xi_{|\alpha|})$ where $a(\xi_1, \ldots, \xi_{|\alpha|})$ is a symmetrical polynomial (see Definition 6). Acting by the same rule, but reserving a set of variables $\zeta_1, \zeta_2, \ldots$ (instead of $\xi_1, \xi_2, \ldots$) for the symbolic representation of v–monomials we get $v^\beta \longmapsto \hat{v}^{|\beta|} b(\zeta_1, \ldots, \zeta_{|\beta|})$. Then

$$u^\alpha v^\beta \longmapsto \hat{u}^{|\alpha|} \hat{v}^{|\beta|} a(\xi_1, \ldots, \xi_{|\alpha|}) b(\zeta_1, \ldots, \zeta_{|\beta|}).$$

Note that the symbol obtained is invariant with respect to the direct product of two permutation groups $\mathcal{S}_{|\alpha|}^\xi \times \mathcal{S}_{|\beta|}^\zeta$.

To the product of two elements $f, g \in \mathcal{R}$ with symbols

$$f \mapsto \hat{u}^n \hat{v}^m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m) \text{ and } g \mapsto \hat{u}^p \hat{v}^q b(\xi_1, \ldots, \xi_p, \zeta_1, \ldots, \zeta_q)$$

corresponds:

$$\begin{aligned} fg \longmapsto \quad & \hat{u}^{n+p} \hat{v}^{m+q} \langle\langle a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m) \\ & b(\xi_{n+1}, \ldots, \xi_{n+p}, \zeta_{m+1}, \ldots, \zeta_{m+q}) \rangle_{\mathcal{S}_{n+p}^\xi} \rangle_{\mathcal{S}_{m+q}^\zeta}, \end{aligned} \quad (20)$$

where the symmetrisation operation is taken with respect to permutations of all arguments ξ and then ζ (the symmetrisation can be made in any order).

If $f \in \mathcal{R}$ has a symbol $f \longmapsto \hat{u}^n \hat{v}^m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)$, then the symbolic representation for the derivative $D_x(f)$ is:

$$D_x(f) \longmapsto \hat{u}^n \hat{v}^m (\xi_1 + \cdots + \xi_n + \zeta_1 + \cdots + \zeta_m) a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m).$$

To the operator D_x we shall assign a special symbol η satisfying the following composition rule (the Leibnitz rule)

$$\begin{aligned} & \eta \circ \hat{u}^n \hat{v}^m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m) \\ = \quad & \hat{u}^n \hat{v}^m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)(\xi_1 + \cdots + \xi_n + \zeta_1 + \cdots + \zeta_m + \eta). \end{aligned}$$

For $f \in \mathcal{R}$ with symbol $f \longmapsto \hat{u}^n \hat{v}^m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m)$ the symbolic representation of the Fréchet derivative (16) is

$$\begin{aligned} f_{*u} &\longmapsto \hat{u}^{n-1} \hat{v}^m n a(\xi_1, \ldots, \xi_{n-1}, \eta, \zeta_1, \ldots, \zeta_m), \\ f_{*v} &\longmapsto \hat{u}^n \hat{v}^{m-1} m a(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_{m-1}, \eta). \end{aligned}$$

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3 Integrability of evolutionary equations

3.1 Study of symmetries of evolutionary equations in symbolic representation

Using the above symbolic representation of the Lie bracket we can study the properties of symmetries in great details.

Theorem 2 *Let the right hand side of evolutionary equation (7) has symbolic representation*

$$F \longmapsto \hat{u}\omega(\xi_1) + \hat{u}^2 a_1(\xi_1, \xi_2) + \hat{u}^3 a_2(\xi_1, \xi_2, \xi_3) + \cdots$$

and the degree of polynomial $\omega(\xi_1)$ is greater than 1. If

$$G \longmapsto \hat{u}\Omega(\xi_1) + \hat{u}^2 A_1(\xi_1, \xi_2) + \hat{u}^3 A_2(\xi_1, \xi_2, \xi_3) + \cdots \tag{21}$$

is a symmetry, then its coefficients can be found recursively

$$A_1(\xi_1, \xi_2) = \frac{G^\Omega(\xi_1, \xi_2)}{G^\omega(\xi_1, \xi_2)} a_1(\xi_1, \xi_2) \tag{22}$$

$$A_{m-1}(\xi_1, ..., \xi_m) = \frac{1}{G^\omega(\xi_1, ..., \xi_m)} \Bigg(G^\Omega(\xi_1, ..., \xi_m) a_{m-1}(\xi_1, ..., \xi_m)$$

$$+ \sum_{j=1}^{m-2} \Bigg\langle (j+1) A_j(\xi_1, ..., \xi_j, \sum_{l=j}^{m-1} \xi_{l+1}) a_{m-1-j}(\xi_{j+1}, ..., \xi_m) \tag{23}$$

$$- (m-j) a_{m-1-j}(\xi_1, ..., \xi_{m-1-j}, \sum_{l=0}^{j} \xi_{m-l}) A_j(\xi_{m-j}, ..., \xi_m) \Bigg\rangle_{\mathcal{S}_m^\xi} \Bigg),$$

where

$$G^\omega(\xi_1, ..., \xi_m) = \omega(\sum_{n=1}^{m} \xi_n) - \sum_{n=1}^{m} \omega(\xi_n). \tag{24}$$

Proof: The proof of the Theorem is straightforward (see, for example [29]). Using (14) we can compute the Lie bracket between F and G. When it vanishes up to $\hat{\mathcal{R}}^2$, we express $A_1(\xi_1, \xi_2)$ from the result, which leads to formula (22). The Lie bracket vanishing up to $\hat{\mathcal{R}}^m$ is equivalent to formula (23). $\square$

Corollary 1 *For the equation stated in Theorem 2, (i). any symmetry has a linear part, that is, $\Omega(\xi_1) \neq 0$; (ii). algebra of symmetries is commutative.*

Proof: (i). Let us assume that $\Omega(\xi_1) = 0$. Then it follows from (22) that $A_1(\xi_1, \xi_2) = 0$. Assuming that $A_k = 0$ for all $1 < k < m - 1$, we get from (23) that $A_{m-1} = 0$. Thus by induction, we get $G = 0$.

(ii). Commutator of two symmetries is a symmetry due to the Jacobi identity, but it does not contain a linear part. Thus it must vanish. $\qquad\square$

Theorem 2 states that a symmetry of equation is uniquely determined by its linear part (i.e. dispersion). For fixed $\Omega(\xi_1)$ all coefficients in the series (21) can be found recursively. Theorem 2 does not mean that any evolutionary equation has a symmetry. The right hand side of (21) must represent a valid symbol, i.e., an element of $\hat{\mathcal{R}}$. Thus:

    a. all coefficients $A_m(\xi_1, \ldots, \xi_{m+1})$ must be polynomial,

    b. it should be a finite number of non-vanishing coefficients A_m.

In general, as it follows from (22) and (23), the coefficients A_k are rational functions – they have denominators G^ω. In order to define a symbol of a symmetry, these denominators must cancel with appropriate factors in the numerators. Thus factorisation properties of polynomials G^ω and G^Ω are crucial for the structure of the symmetry algebra of the equation.

Proposition 3 (F. Beukers [38]) *For any positive integer $m \geq 2$ the polynomial*

$$h_{c,m} = (\xi_1 + \xi_2 + \xi_3 + \xi_4)^m - c_1^{m-1}\xi_1^m - c_2^{m-1}\xi_2^m - c_3^{m-1}\xi_3^m - c_4^{m-1}\xi_4^m,$$

where $\Pi_{i=1}^4 c_i \neq 0$, is irreducible over $\mathbb{C}$.

Proof: Suppose that $h_{c,m} = A \cdot B$ with A, and B two polynomials of positive degree. Then the projective hypersurface Σ given by $h_{c,m} = 0$ consists of two components Σ_A, Σ_B given by $A = 0, B = 0$ respectively. The intersection $\Sigma_A \bigcap \Sigma_B$ consists an infinite number of points, which should be singularities of Σ since

$$\frac{dh_{c,m}}{d\xi_i} = \frac{dA}{d\xi_i} \cdot B \bigg|_{\Sigma_A \bigcap \Sigma_B} + A \cdot \frac{dB}{d\xi_i} \bigg|_{\Sigma_A \bigcap \Sigma_B} = 0.$$

Thus it suffices to show that Σ has finitely many singular points.

We compute the singular points by setting the partial derivatives of $h_{c,m}$ equals to zero, i.e.,

$$\begin{cases} (\xi_1 + \xi_2 + \xi_3 + \xi_4)^{m-1} - (c_1\xi_1)^{m-1} = 0 \\ (\xi_1 + \xi_2 + \xi_3 + \xi_4)^{m-1} - (c_2\xi_2)^{m-1} = 0 \\ (\xi_1 + \xi_2 + \xi_3 + \xi_4)^{m-1} - (c_3\xi_3)^{m-1} = 0 \\ (\xi_1 + \xi_2 + \xi_3 + \xi_4)^{m-1} - (c_4\xi_4)^{m-1} = 0 \end{cases}$$

From these equation follows in particular that

$$\xi_1 = \zeta_1/c_1, \ \xi_2 = \zeta_2/c_2, \ \xi_3 = \zeta_3/c_3, \ \xi_4 = \zeta_4/c_4,$$

where $\zeta_i^{m-1} = 1$ and $\zeta_1/c_1 + \zeta_2/c_2 + \zeta_3/c_3 + \zeta_4/c_4 = 1$. For given $c_i, i = 1 \cdots 4$, we get finitely many singular points. $\qquad\square$

Corollary 2 *Polynomials $G^\omega(\xi_1, \ldots, \xi_n)$, cf. (24) are irreducible for $n \geq 4$.*

Proof: Let $\omega(\xi) = \alpha^m \xi^m + \cdots + \alpha_0$. If $0 \leq m \leq 1$ then G^ω is a constant and therefore irreducible. If $m \geq 2$ polynomial G^ω has the form

$$G^\omega(\xi_1, \ldots, \xi_n) = \alpha_m G^{(m)}(\xi_1, \ldots, \xi_n) + g^\omega,$$

where $\deg(g^\omega) < m$ and

$$G^{(m)}(\xi_1, \ldots, \xi_n) = (\xi_1 + \cdots + \xi_n)^m - \xi_1^m - \cdots - \xi_n^m, \tag{25}$$

which is irreducible according to Proposition 3. $\qquad\square$

Theorem 3 *The algebra of symmetries of the evolutionary equation*

$$u_t = \sum_{k=0}^{n} \alpha_k u_k + f(u_{n-1}, \ldots, u) = F, \quad n \geq 2, \quad \alpha_n \neq 0 \tag{26}$$

where $f(u_{n-1}, \ldots, u) \neq 0$ and

$$f(u_{n-1}, \ldots, u) \in \bigoplus_{m>3} \bigoplus_{p<n} \mathcal{R}_p^m \tag{27}$$

is trivial, i.e., $C_{\mathcal{R}}(F) = \text{Span}_{\mathbb{C}}\{u_1, F\}$.

Proof: In symbolic representation

$$F \longmapsto \hat{u}\omega(\xi_1) + \hat{u}^{m+1} a_m(\xi_1, \ldots, \xi_{m+1}) + \hat{u}^{m+2} a_{m+1}(\xi_1, \ldots, \xi_{m+2}) + \cdots,$$

where $\omega(\xi_1) = \alpha_n \xi_1^n + \cdots + \alpha_1 \xi_1 + \alpha_0$. The condition (27) implies $m \geq 3$ and $\deg(a_m(\xi_1, \ldots, \xi_{m+1})) < n$.

A symmetry of (26) is of the form

$$G \longmapsto \hat{u}\Omega(\xi_1) + \hat{u}^2 A_1(\xi_1, \xi_2) + \hat{u}^3 A_2(\xi_1, \xi_2, \xi_3) + \hat{u}^4 A_3(\xi_1, \xi_2, \xi_3, \xi_4) + \cdots$$

if it exists. We know that its linear part $\Omega(\xi_1) \neq 0$ from Corollary 1. It follows from (23) that $A_k(\xi_1, \ldots, \xi_{k+1}) = 0$ for $k < m$ and

$$A_m(\xi_1, \ldots, \xi_{m+1}) = \frac{G^\Omega(\xi_1, \ldots, \xi_{m+1})}{G^\omega(\xi_1, \ldots, \xi_{m+1})} a_m(\xi_1, \ldots, \xi_{m+1}). \tag{28}$$

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Suppose $\Omega(\xi) \neq \alpha\xi + \beta\omega(\xi)$ for any $\alpha, \beta \in \mathbb{C}$. From Corollary 2, we have that polynomials $G^\Omega(\xi_1, ..., \xi_{m+1})$ and $G^\omega(\xi_1, ..., \xi_{m+1})$ are irreducible and therefore they are co-prime. Since

$$\deg(G^\omega(\xi_1, ..., \xi_{m+1})) = n > \deg(a_m(\xi_1, \ldots, \xi_{m+1})),$$

the right hand side of (28) is a rational function (not a polynomial). Thus there are no symmetries under the assumption.

When $\Omega(\xi) = \alpha\xi + \beta\omega(\xi)$ for some $\alpha, \beta \in \mathbb{C}$, it follows from (23) that $G = \alpha u_1 + \beta F \in \mathrm{Span}_\mathbb{C}\{u_1, F\}$. $\qquad\square$

According to Definition 3, equation (26) is not integrable. In (27) condition $p < n$ is essential. Indeed, equation

$$u_t = u_2 + u^m u_1^2$$

($p = n = 2$) is integrable for any m.

If an evolutionary equation (7) with linear part of order 2 or higher has a nontrivial symmetry, then any approximate symmetry of degree 3 is amendable to any degree. Thus, if we have infinitely many approximate symmetries of degree 3, then we have infinitely many approximate symmetries of arbitrary high degree.

Theorem 4 *Let $\omega(\xi_1)$ be a polynomials of degree greater than 1. Assume that evolutionary equation (7) with linear terms $\hat{u}\omega(\xi_1)$ has a nontrivial symmetry. Then for an approximate symmetry $\sum_{j=1}^3 h_j$, $h_j \in \mathcal{R}^j$ of degree 3, there exists a unique $H = \sum_{j\geq 1} h_j$, $h_j \in \mathcal{R}^j$ such that H is an approximate symmetry of any degree.*

This Theorem is the direct consequence of a more general Theorem 2.3 in [26] (see also Theorem 2.76 p.27 [25]) formulated in the context of filtered Lie modules. According to Theorem 2.3, in application to an evolutionary equation (7) with linear terms $\hat{u}\omega(\xi_1)$, we should require that polynomials $G^\omega(\xi_1, ..., \xi_{m+1})$ and polynomials $G^\Omega(\xi_1, ..., \xi_{m+1})$ defined by (24) have no common factors for some $m > 1$. This is the case for $m = 3$ as it follows from Corollary 2.

The result of Theorem 4 confirms the remark made in [36]:

> *Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalised symmetry implies the existence of infinitely many. (However, this has not been proved in general.)*

For systems of equations and for non-evolutionary equations the conjecture that *the existence of one symmetry implies the existence of (infinitely many) others* has been disproved. In [7] it has shown that example given in [6] is indeed a counterexample to the conjecture (see also [9]). Even a rectified conjecture [39] that *for N-component equations one needs N symmetries* is incorrect either. An example of a system of two equations possessing exactly two nontrivial symmetries is given in [8].

These examples do not contradict to the spirit of our Theorem 4 since they are based upon the nonexistence of approximate symmetries of degree 2, which is one of the conditions in the theorem.

As we have already mentioned above, the existence of a formal recursion operator Λ (12) for an evolutionary equation is a necessary condition for the existence of an infinite hierarchy of symmetries. A similar, but not equivalent, theorem can be stated in the symbolic representation. The difference is in the natural ordering. In the standard representation the coefficients l_k are ordered due to the power of D_x^k in the formal series Λ (12). In the symbolic representation the natural ordering is due to the power of symbol $\hat{u}$. The fact that l_k must be local, i.e. $l_k \in \mathcal{R}$ in the symbolic representation suggests the following definition:

Definition 7 *We say that function $b_m(\xi_1, ..., \xi_m, \eta)$, $m \geq 1$ is k-local if in the expansion as $\eta \to \infty$*

$$b_m(\xi_1, ..., \xi_m, \eta) = \beta_{m1}(\xi_1, ..., \xi_m)\eta^{n_m} + \beta_{m2}(\xi_1, ..., \xi_m)\eta^{n_m-1} + \cdots$$

the first k coefficients $\beta_{ms}(\xi_1, ..., \xi_m)$, $s = 1, ..., k$ are symmetric polynomials in $\xi_1, ..., \xi_m$. We say that $b_m(\xi_1, ..., \xi_m, \eta)$ is local if it is k-local for any k.

Existence of an infinite hierarchy of symmetries implies the existence of a formal recursion operator with local coefficients (Proposition 3 in [29]). Existence of an infinite hierarchy of approximate symmetries of degree N implies that first $N - 1$ coefficients of the formal recursion operator are local. The details of the proof of the following Theorem one can find in [29] (Proposition 3).

Theorem 5 *Suppose equation (7) has an infinite hierarchy of approximate symmetries of degree N*

$$u_{t_i} = \hat{u}\Omega_i(\xi_1) + \sum_{j \geq 1} \hat{u}^{j+1}A_{ij}(\xi_1, \ldots, \xi_{j+1}) = G_i, \quad i = 1, 2, \ldots$$

where $\Omega_i(\xi_1)$ are polynomials of degree m_i and $m_1 < m_2 < \cdots < m_i < \cdots$. Then the coefficients $\phi_m(\xi_1, ..., \xi_m, \eta)$, $m = 1, ..., N-1$ of the formal recursion

operator

$$\Lambda = \eta + \hat{u}\phi_1(\xi_1, \eta) + \hat{u}^2\phi_2(\xi_1, \xi_2, \eta) + \cdots$$

are local.

In symbolic representation equation (13) can be solved [29] in the sense that coefficients of a formal series Λ can be recursively for any evolutionary equation (7):

Theorem 6 *Let $\phi(\eta)$ be an arbitrary function and formal series*

$$\Lambda = \phi(\eta) + \hat{u}\phi_1(\xi_1, \eta) + \hat{u}^2\phi_2(\xi_1, \xi_2, \eta) + \hat{u}^3\phi_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots$$

be a solution of equation (13), then its coefficients $\phi_m(\xi_1, ..., \xi_m, \eta)$ can be found recursively

$$\phi_1(\xi_1, \eta) = \frac{2(\phi(\eta + \xi_1) - \phi(\eta))}{G^\omega(\xi_1, \eta)} a_1(\xi_1, \eta)$$

$$\phi_m(\xi_1, ..., \xi_m, \eta) = \frac{1}{G^\omega(\xi_1, ..., \xi_m, \eta)} \left((m+1)(\phi(\eta + \xi_1 + ... + \xi_m) \right.$$

$$-\phi(\eta))a_m(\xi_1, ..., \xi_m, \eta)$$

$$+ \sum_{n=1}^{m-1} \langle n\phi_n(\xi_1, .., \xi_{n-1}, \xi_n + \cdots + \xi_m, \eta)a_{m-n}(\xi_n, .., \xi_m)$$

$$+ (m-n+1)\phi_n(\xi_1, .., \xi_n, \eta + \sum_{l=n+1}^{m} \xi_l)a_{m-n}(\xi_{n+1}, .., \xi_m, \eta)$$

$$\left. - (m-n+1)a_{m-n}(\xi_{n+1}, .., \xi_m, \eta + \sum_{l=1}^{n} \xi_l)\phi_n(\xi_1, .., \xi_n, \eta)\rangle_{\mathcal{S}_m^\xi} \right).$$

Existence of formal recursion operator with local coefficients is a necessary condition for the existence of an infinite hierarchy of symmetries. It suggests the following test for integrability of equations (7):

- Find a first few coefficients $\phi_n(\xi_1, ..., \xi_n, \eta)$ (first three nontrivial coefficients ϕ_n were sufficient to analyse in all known cases to us).

- Expand these coefficients in series of $1/\eta$

$$\phi_n(\xi_1, ..., \xi_n, \eta) = \sum_{s=s_n} \Phi_{ns}(\xi_1, ..., \xi_n)\eta^{-s}$$

  and check that functions $\Phi_{ns}(\xi_1, ..., \xi_n)$ are polynomials (not rational functions).

This test will be extended and used for non-local and non-evolutionary equations in sections 4.1 and 4.2.

3.2 Global classification of integrable homogeneous evolutionary equations

In this section, we give ultimate global classification of integrable equations of the form

$$u_t = u_n + f(u, \cdots, u_{n-1}), \qquad n \geq 2 \tag{29}$$

where $u_n + f(u, \cdots, u_{n-1})$ is a λ–homogeneous differential polynomial and $\lambda \geq 0$. We give a complete description of integrable equations for all n.

Theorem 4 implies that if equation (29) possesses one higher symmetry and infinitely many approximate symmetries of degree 3, then it possesses infinitely many approximate symmetries of any degree. Therefore to classify integrable equations (29) it suffices to classify equations, which possess infinitely many degree 3 approximate symmetries and then impose the condition of existence of at least one exact symmetry. The classification has been done in the case of λ-homogeneous equations with $\lambda \geq 0$. In the case $\lambda > 0$ see the details in [26], while in the case $\lambda = 0$ the details can be found in [27].

Now we sketch the results for the case $\lambda > 0$ without the detailed proofs. The following statement on factorisation properties of polynomials $G^{(k)}(\xi_1, \ldots, \xi_n)$ (25) plays an important role in the classification of integrable equations:

Theorem 7 $G^{(k)}(\xi_1, \ldots, \xi_n) = t^{(k)} g^{(k)}$, where $(g^{(k)}, g^{(l)}) = 1$ for all $k < l$, and $t^{(k)}$ is one of the following cases.

- $n = 2$:

  - $k = 0 \pmod 2$: $\xi_1 \xi_2$
  - $k = 3 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2)$
  - $k = 5 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2)(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2)$
  - $k = 1 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2)(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2)^2$

- $n = 3$:

  - $k = 0 \pmod 2$: 1
  - $k = 1 \pmod 2$: $(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)$

- $n > 3$: 1

For $n > 3$ the statement follows from more general Theorem 3. For $n = 3$, it has been proven by Beukers and was published in [25, 26] with his kind permission. The case $n = 2$ has a quite remarkable history. In affine co–ordinate $x = \xi_1/\xi_2$ we have $G^{(k)}(\xi_1, \xi_2) = \xi_2^k P_k(x)$ and the problem is reducing to factorisation properties of the Cauchy-Liouville-Mirimanoff polynomials

$$P_k(x) = (1 + x)^k - x^k - 1.$$

The common factors $P_k = x(1+x)^\alpha(1+x+x^2)^\beta g^{(k)}(x)$ and their periodicity have been established in the joint report of Cauchy and Liouville [40]. Using Diophantine approximation theory Beukers has shown that factors $g^{(k)}, g^{(m)}$ are co-prime for $k \neq m$ [41]. Beukers also conjectured that factors $g^{(p)}$ are irreducible over $\mathbb{Q}$. For prime p the irreducibility of $g^{(p)}$ over $\mathbb{Q}$ was earlier conjectured by Mirimanoff [42]. A remarkable progress towards the proof of the Mirimanoff conjecture has been recently achieved in [43].

We now consider λ-homogeneous equations of the form

$$u_t = u_n + f_2 + f_3 + \cdots, \quad f_i \in \mathcal{R}^i \tag{30}$$
$$\longmapsto \hat{u}a_0(\xi_1) + \hat{u}^2 a_1(\xi_1, \xi_2) + \hat{u}^3 a_2(\xi_1, \xi_2, \xi_3) + \cdots,$$

where $n \geq 2$, $\lambda > 0$ and the degree of a polynomial a_j is $n - j\lambda$. Note that if λ is not integer and $i\lambda \notin \mathbb{N}$, then $a_i = 0$. This reduces the number of relevant λ to a finite set.

Let $G \in \mathcal{R}$ be a nontrivial symmetry of (30). Then it is of the form

$$G = u_m + g_2 + g_3 + \cdots, \quad g_i \in \mathcal{R}^i$$
$$\longmapsto \hat{u}A_0(\xi_1) + \hat{u}^2 A_1(\xi_1, \xi_2) + \hat{u}^3 A_2(\xi_1, \xi_2, \xi_3) + \cdots,$$

where $2 \leq m \neq n$ and the degree of polynomial A_j is $m - j\lambda$. For all integers $r \geq 0$ the following formula holds

$$\sum_{i=0}^{r} [\hat{u}^{i+1} a_i, \hat{u}^{r-i+1} A_{r-i}] = 0. \tag{31}$$

Clearly we have $[\hat{u}a_0, \hat{u}A_0] = 0$. The next equation to be solved is $[\hat{u}a_0, \hat{u}^2 A_1] + [\hat{u}^2 a_1, \hat{u}A_0] = 0$, which is trivially satisfied if equation (30) has no quadratic terms: $f_2 = 0$. Let us concentrate on the case $f_2 \neq 0$. In this case, using Theorem 4, we see that the existence of a symmetry is uniquely determined by the existence of its quadratic term [25, 26].

We now make a very interesting observation. Assume n and q are both odd. Let us compute the symmetry of equation (30) with linear term u_q. Its quadratic terms, cf. (31), have the following symbolic expression

$$\frac{a_1 \; (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)^{s-s'} \; g^{(q)}(\xi_1, \xi_2)}{g^{(n)}(\xi_1, \xi_2)}. \tag{32}$$

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Proposition 7 implies that $\lambda \le 3 + 2\min(s, s')$, where $s' = \frac{n+3}{2}$ (mod 3) and $s = \frac{q+3}{2}$ (mod 3). We see that if expression (32) is a polynomial, then it defines a symmetry $Q = u_q + Q_2 + \cdots$ since Q is determined by its quadratic term Q_2. The evolutionary equations defined by Q has the same symmetries as equation (30). So instead of (30) we may consider the equation given by Q. The lowest possible q is $2s + 3$ for $s = 0, 1, 2$. Therefore we only need to consider λ-homogeneous equations with $\lambda \le 7$) of orders ≤ 7.

A similar observation can be made for even $n > 2$. Suppose we have found a nontrivial symmetry with quadratic term

$$\frac{a_1 \, G^{(q)}}{\xi_1 \xi_2 \, g^{(n)}}$$

This immediately implies $\lambda \le 2$. Then the quadratic term $2\frac{a_1}{g^{(n)}}$ defines a symmetry Q starting with u_2. Therefore, we only need to find the symmetries of equations of order 2 to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations (with $\lambda \le 2$) starting with an even linear term.

Finally, we analyse the case when equation (30) has no quadratic terms. If $a_i = 0$ for $i = 1, \cdots, j - 1$, then we look at the equation $[\hat{u}a_0, \hat{u}^{j+1}A_j] + [\hat{u}^{j+1}a_j, \hat{u}A_0] = 0$, i.e.

$$A_j = \frac{G^{(m)}(\xi_1, \cdots, \xi_{j+1}) \, a_j}{G^{(n)}(\xi_1, \cdots, \xi_{j+1})}.$$

From Proposition 7 and the proof of Theorem 3, we know there are no symmetries for the equation when $j \ge 3$, or when $j = 2$ and n is even. When $j = 2$ and n is odd, it can only have odd order symmetries. In this case one can remark that if the equation possesses symmetries for any m then it must possess a symmetry of order 3.

By now, we have proved the following

Theorem 8 *A nontrivial symmetry of a λ-homogeneous equation with $\lambda > 0$ is part of a hierarchy starting at order* 2, 3, 5 *or* 7.

Only an equation with nonzero quadratic or cubic terms can have a nontrivial symmetry. For each possible $\lambda > 0$, we must find a third order symmetry for a second order equation, a fifth order symmetry for a third order equation, a seventh order symmetry for a fifth order equation with quadratic terms, and the thirteenth order symmetry for a seventh order equation with quadratic terms. The last case can be easily reduced to the case of fifth order equations by determining the quadratic terms of the equation. The details of this final computation are described in [44].

Theorem 9 *Let $\lambda > 0$. Suppose that a λ–homogeneous polynomial evolutionary equation*

$$u_t = u_n + f(u, \cdots, u_{n-1}), \qquad n \geq 2$$

possesses nontrivial symmetries. Then it is a symmetry of one of the following equations up to a transformations $u \mapsto \alpha u$, $\alpha \in \mathbb{C}$:

Burgers equation

$$u_t = u_2 + uu_1$$

Korteweg–de Vries equation

$$u_t = u_3 + uu_1$$

Potential Korteweg–de Vries equation

$$u_t = u_3 + u_1^2$$

Modified Korteweg–de Vries equation

$$u_t = u_3 + u^2 u_1$$

Ibragimov-Shabat equation

$$u_t = u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1$$

Kaup-Kupershmidt equation

$$u_t = u_5 + 10uu_3 + 25u_1 u_2 + 20u^2 u_1$$

Potential Kaup-Kupershmidt equation

$$u_t = u_5 + 10u_1 u_3 + \frac{15}{2} u_2^2 + \frac{20}{3} u_1^3$$

Sawada-Kotera equation

$$u_t = u_5 + 10uu_3 + 10u_1 u_2 + 20u^2 u_1$$

Potential Sawada-Kotera equation

$$u_t = u_5 + 10u_1 u_3 + \frac{20}{3} u_1^3$$

Kupershmidt Equation

$$u_t = u_5 + 5u_1 u_3 + 5u_2^2 - 5u^2 u_3 - 20\, uu_1 u_2 - 5u_1^3 + 5u^4 u_1$$

Finally we note that all the consideration in this section can be extended to the case when the dependent variable u takes its values in some associative non-commutative algebra (such as matrix, operator, Clifford, and group algebras). A complete classification for $\lambda > 0$ homogeneous equations with linear leading term in the case of non-commutative free associative algebra was carried out in [28].

4   Classification results for Non-local integrable equations

The perturbative symmetry approach in the symbolic representation allows to derive integrability conditions for certain type of non-local equations. In this section we consider two types of such equations: the Benjamin-Ono type and the Camassa-Holm type [29, 32].

Benjamin-Ono equation reads

$$u_t = H(u_2) + 2uu_1, \tag{33}$$

where H denotes the Hilbert transform

$$H(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{y - x} dy.$$

It is well known that the higher symmetries and conservation laws of the Benjamin-Ono equation contain nested Hilbert transform and thus an appropriate extension of the differential ring $\mathcal{R}$ is required. The construction of such extension is similar to the one proposed by Mikhailov and Yamilov in [45] for $2 + 1$ dimensional equations (see also section 6).

The second example is Camassa-Holm type equation:

$$m_t = cmu_1 + um_1, \quad m = u - u_2, \quad c \in \mathbb{C} \setminus \{0\}. \tag{34}$$

This equation is known to be integrable for $c = 2$ [46] and for $c = 3$ [47]. Equation (34) is not in the evolutionary form, but if we exclude one of the dependent variables (say u) then we obtain a non-local equation

$$m_t = cm\Delta(m_1) + \Delta(m)m_1, \quad \Delta = (1 - D_x^2)^{-1} \tag{35}$$

and again the ring extension is required.

The symbolic representation and the concept of quasi-locality [45] are the key ideas in the extension of the symmetry approach to non-local equations. In the definitions of all basic objects such as symmetries, formal recursion operators, conservation laws etc. we replace the ring of differential polynomials $\mathcal{R}$ by an appropriately extended ring. Elements of this extended ring we call quasi-local polynomials (see details in the next section). Symbolic representation gives us a simple criteria to decide if a given expression belong to the extended ring. In this extended setting Theorem 2 and most of the results of Section 3.1 hold if we just replace "local" by "quasi-local" in conditions and statements.

4.1 Benjamin-Ono type equations and Intermediate long wave equation

Let us consider the following sequence of ring extensions:

$$\mathcal{R}_{H^0} = \mathcal{R}\,, \quad \mathcal{R}_{H^{n+1}} = \overline{\mathcal{R}_{H^n} \bigcup H(\mathcal{R}_{H^n})}\,,$$

where the set $H(\mathcal{R}_{H^n})$ is defined as $H(\mathcal{R}_{H^n}) = \{H(a); a \in \mathcal{R}_{H^n}\}$ and the horizontal line denotes the ring closure. Each $\mathcal{R}_{H^n}$ is a ring and the index n indicates the nesting depth of the operator H:

$$\mathcal{R}_{H^0} \subset \mathcal{R}_{H^1} \subset \mathcal{R}_{H^2} \subset \cdots \subset \mathcal{R}_{H^n} \subset \cdots \subset \mathcal{R}_{H^\infty} = \mathcal{R}_H.$$

Elements of $\mathcal{R}_{H^n}$, $n \geq 1$ we call *quasi–local polynomials*. The right hand side of equation (33), its symmetries and densities of conservation laws are quasi–local polynomials.

We now consider scalar evolutionary equations, whose right hand side is a quasi–local polynomial

$$u_t = F, \quad F \in \mathcal{R}_H. \tag{36}$$

For the definition of its symmetry we replace $\mathcal{R}$ by $\mathcal{R}_H$ in Definition 2. Actual computations in $\mathcal{R}_H$ lead to quite cumbersome calculations. On the other hand, in the symbolic representation computations simplify drastically and results can be neatly formulated.

In the symbolic representation operator H is represented by $i\,\mathrm{sign}(\eta)$. So the symbolic representation of the ring extensions is obvious. Suppose $f \in \mathcal{R}_{H^0}$ and

$$f \mapsto \hat{u}^n a(\xi_1, \ldots, \xi_n).$$

Then

$$H(f) \mapsto \hat{u}^n i\,\mathrm{sign}(\xi_1 + \cdots + \xi_n) a(\xi_1, \ldots, \xi_n).$$

In the extended ring all the definitions, such as the Fréchet derivative, Lie bracket and approximate symmetries, are exactly the same as in the local case. However, the symbols of elements of the extended ring are symmetric sign-polynomials instead of symmetric polynomials. For example, the symbolic representation of $H(u_n)$ and $H(uH(u_1))$ is:

$$H(u_n) \mapsto \hat{u}\, i\,\mathrm{sign}(\xi_1)\xi_1^n,$$
$$H(uH(u_1)) \mapsto -\frac{\hat{u}^2}{2}\,\mathrm{sign}(\xi_1 + \xi_2)\left(\xi_1\,\mathrm{sign}(\xi_1) + \xi_2\,\mathrm{sign}(\xi_2)\right).$$

The symbolic representation of the Benjamin-Ono equation (33) is

$$u_t = i\hat{u}\, \text{sign}(\xi_1)\xi_1^2 + \hat{u}^2(\xi_1 + \xi_2).$$

Counting the degrees of sign-polynomials we assume that $\deg(\,\text{sign}(\xi_1 + \cdots + \xi_k)) = 0$.

Theorem 2 for evolutionary equations and symmetries in $\mathcal{R}_H$, in the symbolic representation remains the same with the only amendment that all symbols now are sign–polynomials. To introduce a formal recursion operator for equation (36) we introduce a notion of asymptotically local functions, which generalises the notion of local functions, cf. Definition 7:

Definition 8 *A function $a_n(\xi_1, \ldots, \xi_n, \eta)$ is called asymptotically local if the coefficients $a_{np}(\xi_1, \ldots, \xi_n)$ and $\tilde{a}_{np}(\xi_1, \ldots, \xi_n)$ of its expansion at $\eta \to \infty$:*

$$a_n(\xi_1, .., \xi_n, \eta) = \sum_{p=s_n}^{\infty} a_{np}(\xi_1, .., \xi_n)\eta^{-p} + \sum_{p=\tilde{s}_n}^{\infty} \tilde{a}_{np}(\xi_1, .., \xi_n)\, sign(\eta)\eta^{-p}$$

are sign–polynomials, i.e. represent elements from the extended ring $\mathcal{R}_H$.

In the above expansion we take into account $\text{sign}(\eta + \sum_j \xi_j) = \text{sign}(\eta)$ as $\eta \to \infty$.

Definition 9 *A formal series*

$$\Lambda = \phi(\eta) + \hat{u}\phi_1(\xi_1, \eta) + \hat{u}^2\phi_2(\xi_1, \xi_2, \eta) + \hat{u}^3\phi_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots$$

is called a formal recursion operator of equation (36) if it satisfies equation (13) and all its coefficients are asymptotically local.

Without loss of generality function $\phi(\eta)$ can be chosen as either $\phi(\eta) = \eta$ or $\phi(\eta) = \eta\, \text{sign}(\eta)$.

As in the local case, we can solve equation (13) with respect to coefficients of the formal recursion operator and Proposition 6 holds. The generalisation of Theorem 5 is straightforward, we just replace "local" by asymptotically local.

For the Benjamin-Ono equation (33), the first coefficient $\phi_1(\xi_1, \eta)$ of corresponding formal recursion operator

$$\Lambda = \eta + u\phi_1(\xi_1, \eta) + u^2\phi_2(\xi_1, \xi_2, \eta) + \cdots$$

looks like

$$\phi_1(\xi_1, \eta) = \text{sign}(\eta) + \frac{\xi_1(\,\text{sign}(\xi_1) + \,\text{sign}(\eta))}{2\eta} + O(\frac{1}{\eta^7})$$

and it is asymptotically local. One may easily check asymptotic locality of other coefficients $\phi_2(\xi_1, \xi_2, \eta)$, $\phi_3(\xi_1, \xi_2, \xi_3, \eta)$, ...

In this setting we can classify the generalisation of Benjamin-Ono type equation. Consider an equation of the form

$$
\begin{aligned}
u_t = {} & H(u_2) + c_1 u u_1 + c_2 H(u u_1) + c_3 u H(u_1) + c_4 u_1 H(u) + \\
& + c_5 H(u H(u_1)) + c_6 H(u) H(u_1),
\end{aligned} \tag{37}
$$

where c_j are complex constants. The linear term of this equation coincides with the linear term of the Benjamin-Ono equation and all possible homogeneous terms are included if we suppose that H is a zero-weighed operator $W(H(f)) = W(f)$ and weight of the variable u equals 2. We also take into account that $H^2 = -1$ and the Hilbert-Leibnitz rule

$$
H(fg) = f H(g) + g H(f) + H(H(f) H(g)).
$$

The following theorem holds (see the details and proof in [32]).

Theorem 10 *Equation of the form (37) possesses an infinite hierarchy of higher symmetries if and only if it is, up to the point transformation $u \mapsto au + bH(u)$, $a^2 + b^2 \neq 0$ and re-scalings $x \mapsto \alpha x$, $t \mapsto \beta t$, $a, b, \alpha, \beta \in \mathbb{C}$, one of the list*

$$
u_t = H(u_2) + D_x(\frac{1}{2} c_1 u^2 + c_2 u H(u) - \frac{1}{2} c_1 (u)^2); \tag{38}
$$

$$
u_t = H(u_2) + D_x(\frac{1}{2} c_1 u^2 + \frac{1}{2} c_2 H(u^2) - c_2 u H(u)); \tag{39}
$$

$$
\begin{aligned}
u_t = {} & H(u_2) + i u u_1 \pm H(u u_1) \mp u H(u_1) \mp 2 u_1 H(u) \\
& - i H(u H(u_1));
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
u_t = {} & H(u_2) + H(u u_1) + u_1 H(u) \pm i H(u H(u_1)) \\
& \pm i H(u) H(u_1).
\end{aligned} \tag{41}
$$

The proof of this theorem requires to check the quasi–locality of the first three coefficients of the corresponding formal recursion operators. Equations (38), (40) and (41) can be reduced to the Burgers equation. In the case $c_1^2 + c_2^2 \neq 0$ equation (39) can be transformed into the Benjamin-Ono equation (33). When $c_1^2 + c_2^2 = 0$, it is equivalent to the Burgers equation. The explicit form of transformations are in [32]. The properties of the Benjamin-Ono equation have been studied in [2].

Finally we draw our attention to the intermediate long wave equation

$$
u_t = -\delta^{-1} u_1 + 2 u u_1 + \mathcal{T}(u_2),
$$

where δ is a real constant parameter and

$$\mathcal{T}(u(x)) := \frac{1}{2\delta} \int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(x-y)\right) u(y) dy \ .$$

This equation was derived by Joseph [48] as the equation describing propagation of non-linear waves in a fluid of finite depth. The intermediate long wave equation is an intermediate between Benjamin-Ono and the Korteweg–de Vries equations in the sense that the limit $\delta \to \infty$ yields Benjamin-Ono equation, while $\delta \to 0$ gives KdV equation. The intermediate long wave equation possesses an infinite hierarchy of higher symmetries and is integrable by the inverse scattering method [49]. As in the case of the Benjamin-Ono equation, all its higher symmetries contain nested $\mathcal{T}$ operator.

We consider the general non-linear equation of the intermediate long wave form with some linear operator $\mathcal{T}$

$$u_t = \mathcal{T}(u_2) + 2uu_1 \tag{42}$$

and address the question: for which linear operators $\mathcal{T}$ this equation possesses an infinite hierarchy of higher symmetries/conservation laws? In [49] Ablowitz *et al* have shown that if equation (42) possesses infinitely many conservation laws then the conditions

$$\mathcal{T}(u\mathcal{T}v + v\mathcal{T}u) = (\mathcal{T}u)(\mathcal{T}v) - uv, \tag{43}$$

$$\int_{-\infty}^{\infty} (u\mathcal{T}v + v\mathcal{T}u) dx = 0$$

must be satisfied.

The perturbative symmetry approach allows to derive conditions for the operator $\mathcal{T}$ necessary for the existence of an infinite hierarchy of higher symmetries. All the steps are similar to the case of Benjamin-Ono type equation:

- Extend the differential ring by the operator $\mathcal{T}$ exactly in the same way as we did with H and define $\mathcal{R}_{\mathcal{T}}$ and its symbolic representation.

- Define higher symmetries in the extended ring $\mathcal{R}_{\mathcal{T}}$.

- Introduce a formal recursion operator and asymptotic locality of its coefficients.

Without going into the details (see [29] and [50]) we present the following statement:

Theorem 11 *Assume that the operator $\mathcal{T}$ has the symbolic representation $if(k)$:*

$$\mathcal{T}(u(x)) = i \int_{-\infty}^{\infty} f(k)\hat{u}(k)e^{ikx}dk$$

and that $f(k) \to 1$, faster than any power of k^{-1}, as $k \to +\infty$. Then if equation (42) possesses a formal recursion operator with first three coefficients being asymptotically local then $f(k)$ satisfies the functional equation:

$$f(x + y)\left(f(x) + f(y)\right) = f(x)f(y) + 1 \tag{44}$$

Formula (44) is equivalent to (43) in the symbolic representation. Its general odd solution, smooth on the real line except the origin, is given by

$$f(k) = \coth(\delta k)$$

which corresponds to the intermediate long wave equation and the limiting case $\delta \to +\infty$ corresponds the Benjamin-Ono equation

$$f(k) = \text{sign}(k).$$

These are the only such equations possessing infinitely many conservation laws. The only even solution of (44) is

$$f(k) = const$$

leading to the Burgers equation (up to a re-scaling), which has no non-trivial conservation laws.

4.2   Camassa-Holm type equations

We now apply the perturbative symmetry approach to determine integrable cases of the Camassa-Holm type equation (35). It contains operator $\Delta = (1 - D_x^2)^{-1}$ and therefore we extend the differential ring $\mathcal{R}$ by operator Δ and define the Δ–extended ring $\mathcal{R}_\Delta$ as we did for $\mathcal{R}_H$ in section 4.1.

The symbolic representation of the operator Δ is $\Delta \mapsto \frac{1}{1-\eta^2}$. Therefore if $f \in \mathcal{R}$ with symbol $\hat{u}^n a(\xi_1, \ldots, \xi_n)$ then $\Delta(f)$ has the symbol $\hat{u}^n \frac{a(\xi_1,\ldots,\xi_n)}{1-(\xi_1+\cdots+\xi_n)^2}$.

All the definitions remain the same as in the local case with the amendment $\mathcal{R} \to \mathcal{R}_\Delta$.

We consider a formal recursion operator for the equation (35). First of all we introduce a linear term to the equation (35) by the change of variable $m \mapsto m + 1$ (note that $\Delta(1) = 1$) and consider the equation:

$$m_t = c\Delta(m_1) + cm\Delta(m_1) + \Delta(m)m_1 + m_1 := F. \tag{45}$$

Its symbolic representation reads:

$$F \mapsto \hat{m}\omega(\xi_1) + \hat{m}^2 a(\xi_1, \xi_2),$$

where $\omega(\xi_1) = \frac{c\xi_1}{1-\xi_1^2} + \xi_1, \quad a(\xi_1, \xi_2) = \frac{c\xi_1+\xi_2}{2(1-\xi_1^2)} + \frac{c\xi_2+\xi_1}{2(1-\xi_2^2)}$. The following statement holds [32]:

Theorem 12 *The first two coefficients of the formal recursion operator*

$$\Lambda = \eta + \hat{m}\phi_1(\xi_1, \eta) + \hat{m}^2\phi_2(\xi_1, \xi_2, \eta) + \cdots$$

for the equation (45) are quasi-local if and only if $c = 2$ or $c = 3$.

The case $c = 2$ corresponds to the Camassa-Holm equation, while the case $c = 3$ corresponds to the Degasperis-Processi equation. In fact, one can show [51] that equation (45) (or equation (35)) with $c = 2$ or $c = 3$ possesses an infinite dimensional algebra of *local* higher symmetries in variable m even if the equation is non-local.

5 Integrable Boussinesq type equations

In this section we give a brief account of our results (see details in [30, 31]) on integrable systems of the form:

$$\begin{cases} u_t = v_r, \\ v_t = \alpha u_{p-r} + \beta v_q + F(u, u_1, ..., u_{p-r-1}, v, v_1, ..., v_{q-1}), \end{cases} \tag{46}$$

where $p > q \geq r \geq 0$, $\alpha, \beta \in \mathbb{C}$. System (46) can be reduced to a single second order (in time) non-evolutionary equation of order p:

$$w_{tt} = \alpha w_p + \beta w_{q,t} + F(w_r, w_{r+1}, \ldots, w_{p-1}, w_t, w_{1,t}, \ldots, w_{q-1,t})$$

in the variable w, such that $v = \partial_t w$, $u = w_r$ and w_k denotes $\partial_x^k w$. If function F does not depend on $v, v_1, \ldots, v_{r-1}$ then one can eliminate v from the second equation and rewrite the system in the form

$$u_{tt} = \alpha u_p + \beta u_{t,q} + K(u, u_1, u_2, ..., u_{p-1}, u_t, u_{t,1}, u_{t,2}, ..., u_{t,q-1}), \tag{47}$$

where $K = D_x^r(F)$.

The famous integrable Boussinesq equation [52]

$$u_{tt} = u_{xxxx} + (u^2)_{xx} \tag{48}$$

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belongs to this class. Recently all integrable equations of the form

$$u_{tt} = u_{xxx} + F(u, u_x, u_{xx}, u_t, u_{tx})$$

have been classified and comprehensively studied in [53]. In particularly, it has been shown that equation

$$w_{tt} = w_{xxx} + 3w_x w_{tx} + w_{xx} w_t - 3w_x^2 w_{xx}.$$

is integrable.

Sixth order $(p = 6)$ integrable equations of the form (47) can be obtained as reductions of the Sato hierarchies corresponding to KP, BKP and CKP equations [54, 55] as well as derived from equations studied by Drinfeld and Sokolov [56, 57].

A non-evolutionary equation

$$u_{tt} = K(u, u_x, u_{xx}, \cdots, \partial_x^n u, u_t, u_{tx}, u_{txx}, \cdots, \partial_x^m u_t), \qquad (49)$$

can always be replaced by a system of two evolutionary equations

$$\begin{cases} u_t = v, \\ v_t = K(u, u_x, u_{xx}, ..., \partial_x^n u, v, v_x, v_{xx}, ..., \partial_x^m v). \end{cases}$$

Non-evolutionary equation (49) may have other representations in the evolutionary form. If $K = D_x(G)$, then the system of evolutionary equations

$$u_t = v_x, \qquad v_t = G.$$

also represents (49).

For example, eliminating variable v from the following systems

Case 1:

$$u_t = v, \qquad v_t = u_{xxxx} + (u^2)_{xx}, \qquad (50)$$

Case 2:

$$u_t = v_x, \qquad v_t = u_{xxx} + (u^2)_x, \qquad (51)$$

Case 3:

$$u_t = v_{xx}, \qquad v_t = u_{xx} + u^2, \qquad (52)$$

we obtain the same Boussinesq equation (48) on variable u.

There is a subtle, but important difference among the above representations of the Boussinesq equation (48). The system (50) has only a finite number of local infinitesimal symmetries (i.e. symmetries whose generators can be expressed in terms of u, v and a finite number of their derivatives) while equations (51) and (52) have infinite hierarchies of local symmetries.

The reason is simple – infinitesimal symmetries of equation (51), which depend on the variable v, cannot be expressed in terms of u and its derivatives, since formally $v = D_x^{-1} u_t$. Equation (51) possesses only a finite number of symmetries that do not depend explicitly on the variable v.

Homogeneous equations play central role in the theory of integrable equations.

Proposition 4 *Let system (46) be* **w***–homogeneous of weight* μ*, then* $\mathbf{w} = (w_u, w_u + \mu - r)$ *and*

i. if p is even, $p = 2n$, $n \in \mathbb{N}$, then $\mu = q = n$,

ii. if p is odd, $p = 2n + 1$, $n \in \mathbb{N}$, then $\beta = 0$, $\mu = p/2$, $q = n + 1$.

System (46) we will write in the vector form

$$\mathbf{u}_t = \mathbf{F}, \quad \mathbf{F} = (v_r, f)^T. \tag{53}$$

Suppose $\mathbf{G} = (g, h)^T$ is a generator of a symmetry, then

$$[\mathbf{F}, \mathbf{G}] = 0 \iff D_x^r(h) = D_{\mathbf{F}}(g), \quad D_{\mathbf{G}}(f) = D_{\mathbf{F}}(h).$$

It follows from the first equation that $D_{\mathbf{F}}(g)$ belongs to the image of D_x^r and thus the second component of the symmetry generator $\mathbf{G}$ can be expressed as $h = D_x^{-r}(D_{\mathbf{F}}(g))$ and the symmetry is completely defined by its first component. Substitution into the second equation yields

$$D_{\mathbf{F}}^2(g) - D_x^r D_{\mathbf{G}}(f) = 0 \,.$$

For approximate symmetry of degree n we obviously get

$$D_{\mathbf{F}}^2(g) - D_x^r D_{\mathbf{G}}(f) = o(\mathcal{R}^n) \,.$$

Using the above observation for systems (46) we can restrict the action of the recursion operator to the first component of symmetries.

For example, if we represent the Boussinesq equation (48) in the form of evolutionary system (51), then

$$\mathbf{F}_* = \begin{pmatrix} 0 & D_x \\ D_x^3 + 2u D_x + 2u_1 & 0 \end{pmatrix}$$

and it is easy to verify that a pseudo-differential operator

$$\Lambda = \begin{pmatrix} 3v + 2v_1 D_x^{-1} & 4D_x^2 + 2u + u_1 D_x^{-1} \\ \Lambda_{21} & 3v + v_1 D_x^{-1} \end{pmatrix},$$

where $\Lambda_{21} = 4D_x^4 + 10uD_x^2 + 15u_1D_x + 9u_2 + 4u^2 + (2u_3 + 4uu_1)D_x^{-1}$, satisfies equation (19) and therefore is a recursion operator. We can restrict the action of the recursion operator on the first component. If g_1 is the first component of a symmetry of the Boussinesq equation (51), then

$$g_2 = \Re(S) = \left(3v + 2v_1D_x^{-1} + (4D_x^2 + 2u + u_1D_x^{-1})D_tD_x^{-1}\right)(g_1)$$

is the first component of the next symmetry in the hierarchy. We call $\Re$ a restricted recursion operator.

A space-shift, generated by $\mathbf{u}_1 = (u_1, v_1)^{\mathrm{tr}}$, is a symmetry of the Boussinesq equation (51). Taking u_1 as a seed, we can construct an infinite hierarchy $g_{3k+1} = \Re^k(u_1)$ of symmetries of weights $3k+1$, $k = 0, 1, 2, \ldots$. For example,

$$g_4 = 4v_3 + 4v_1u + 4vu_1 = 4D_x(v_2 + vu).$$

We see that g_4 is a total derivative and therefore $g_7 = \Re(g_4) \in \mathcal{R}$ is the next symmetry in the hierarchy, etc. The Boussinesq equation itself is not a member of this hierarchy. If we take a seed symmetry, corresponding to the time-translation $(v_1, u_3 + 2uu_1)^{\mathrm{tr}}$ we obtain another infinite hierarchy of symmetries $g_{3k+2} = \Re^k(v_1)$, $k = 0, 1, 2, \ldots$. The Boussinesq equation does not have symmetries of weight $3k$, $k \in \mathbb{N}$. One can show that g_{3k+1} and g_{3k+2} are elements of the ring $\mathcal{R}$ for any $k \in \mathbb{N}$ and therefore $\Re$ generates two infinite hierarchies of symmetries of the Boussinesq equation. Moreover, all symmetries from the both hierarchies commute with each other.

In general, for system (53) a recursion operator is completely determined by its two entries Λ_{11} and Λ_{12} (the first row of Λ). The restricted recursion operator $\Re$ for system (53) can be represented as

$$\Re = \Lambda_{11} + \Lambda_{12}D_x^{-r}D_t.$$

In symbolic representation system (46) takes the form

$$\begin{cases} \hat{u}_t = \hat{v}\zeta_1^r, \\ \hat{v}_t = \alpha\hat{u}\xi_1^{p-r} + \beta\hat{v}\zeta_1^q \\ \quad + \sum_{k\geq 2}\sum_{i=0}^{k}\hat{u}^i\hat{v}^{k-i}a_{i,k-i}(\xi_1, .., \xi_i, \zeta_1, .., \zeta_{k-i}). \end{cases} \tag{54}$$

It follows from Proposition 4 that for $\mathbf{w}$–homogeneous systems (46) $q = p/2$ (for even p) and coefficients $a_{i,j}(\xi_1, \ldots, \xi_i, \zeta_1, \ldots, \zeta_j)$ are homogeneous polynomials of degree

$$\deg(a_{i,j}) = p + (j-1)r - (i+j-1)w_u + (j-1)r - \frac{1}{2}jp,$$

in variables $\xi_1, \ldots, \xi_i, \zeta_1, \ldots, \zeta_j$, they are symmetric in $\xi_1, \ldots, \xi_i$ and in $\zeta_1, \ldots, \zeta_j$. If $\deg(a_{i,j})$ is not a non-negative integer, then $a_{i,j} = 0$. We shall assume that the weight $w_u > 0$ and thus the sum in (54) is finite due to only a finite number of non-zero coefficients a_{ij}.

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5.1 Even order equations

In this section we study homogeneous (with a positive weight $w_u > 0$) even order ($p = 2n$) equations (46) assuming $r = 1$.

In symbolic representation such equations take of the form

$$\begin{cases} \hat{u}_t = \hat{v}\zeta_1 \\ \hat{v}_t = \hat{u}\omega_1(\xi_1) + \hat{v}\omega_2(\zeta_1) + \hat{u}^2 a_{20}(\xi_1, \xi_2) + \hat{u}\hat{v}a_{11}(\xi_1, \zeta_1) + \\ \quad + \hat{v}^2 a_{02}(\zeta_1, \zeta_2) + \hat{u}^3 a_{30}(\xi_1, \xi_2, \xi_3) + \cdots = \hat{f}, \end{cases} \tag{55}$$

where $\omega_1(\xi_1) = \alpha\xi_1^{2n-1}$ and $\omega_2(\zeta_1) = \beta\zeta_1^n$ and without loss of generality we shall represent α in the form

$$\alpha = \frac{\mu^2 - \beta^2}{4}$$

and use parameters μ, β instead of α, β.

Symmetries of system (55) are determined by their first component, which in symbolic representation can be written in the form

$$\hat{u}_\tau = \hat{u}\Omega_1(\xi_1) + \hat{v}\Omega_2(\zeta_1) + \hat{u}^2 A_{20}(\xi_1, \xi_2) + \hat{u}\hat{v}A_{11}(\xi_1, \zeta_1) +$$
$$+ \hat{v}^2 A_{02}(\zeta_1, \zeta_2) + \hat{u}^3 A_{30}(\xi_1, \xi_2, \xi_3) + \cdots,$$

where $\Omega_1(\xi_1)$, $\Omega_2(\zeta_1)$, A_{ij} are polynomials. For fixed $\Omega_1(\xi_1)$, $\Omega_2(\zeta_1)$ the coefficients A_{ij} can be found recursively (there is a generalisation of Theorem 2 to the case of many dependent variables [31]).

The Fréchet derivative $\mathbf{F}_*$ of the system (53)with $r = 1$ in the symbolic representation has the form

$$\hat{\mathbf{F}}_* = \begin{pmatrix} 0 & \eta \\ \hat{f}_{*,u} & \hat{f}_{*,v} \end{pmatrix}$$

where

$$\hat{f}_{*,u} = \omega_1(\eta) + 2\hat{u}a_{20}(\xi_1, \eta) + \hat{v}a_{11}(\eta, \zeta_1) + 3\hat{u}^2 a_{30}(\xi_1, \xi_2, \eta) + \cdots$$
$$\hat{f}_{*,v} = \omega_2(\eta) + \hat{u}a_{11}(\xi_1, \eta) + 2\hat{v}a_{02}(\zeta_1, \eta) + \hat{u}^2 a_{21}(\xi_1, \xi_2, \eta) + \cdots$$

Formal recursion operator can be defined as a 2×2 matrix $\hat{R}$ whose entries are formal series

$$\hat{R}_{11} = \phi_{00}(\eta) + \hat{u}\phi_{10}(\xi_1, \eta) + \frac{1}{2}\hat{v}\phi_{01}(\zeta_1, \eta) + \hat{u}^2 \phi_{20}(\xi_1, \xi_2, \eta) +$$
$$+ \hat{u}\hat{v}\phi_{11}(\xi_1, \zeta_1, \eta) + \hat{v}^2 \phi_{02}(\zeta_1, \zeta_2, \eta) + \cdots$$
$$\hat{R}_{12} = \psi_{00}(\eta) + \frac{1}{2}\hat{u}\psi_{10}(\xi_1, \eta) + \hat{v}\psi_{01}(\zeta_1, \eta) + \hat{u}^2 \psi_{20}(\xi_1, \xi_2, \eta) +$$
$$+ \hat{u}\hat{v}\psi_{11}(\xi_1, \zeta_1, \eta) + \hat{v}^2 \psi_{02}(\zeta_1, \zeta_2, \eta) + \cdots,$$

and

$$\hat{R}_{21} = \eta^{-1} \circ (\hat{R}_{11,t} + \hat{R}_{12} \circ \hat{f}_{*,u}), \quad \hat{R}_{22} = \eta^{-1} \circ (\hat{R}_{12,t} + \hat{R}_{12} \circ \hat{f}_{*,v} + \hat{R}_{11} \circ \eta).$$

satisfying equation

$$\hat{R}_t = [\hat{\mathbf{F}}_*, \hat{R}]$$

and all the coefficients $\phi_{ij}, \psi_{ij}, i, j = 0, 1, 2, \ldots$ of the formal series $\hat{R}_{11}$ and $\hat{R}_{12}$ are local. Approximate recursion operator of degree k can be viewed as a truncation of $\hat{R}$ (terms with $\hat{u}^i \hat{v}^j, i + j > k$ are omitted or ignored), so the existence of approximate recursion operator is a necessary condition for the existence of a formal recursion operator and as well necessary condition for the existence of an infinite hierarchy of symmetries of equation (55). For fixed $\phi_{00}(\eta), \psi_{00}(\eta)$ the coefficients of a formal recursion operator can be found recursively, similar to the case of one dependent variable (Theorem 6). The property of locality of the coefficients imposes constraints on the coefficients of equation (55) and eventually leads to the isolation of integrable systems (see details in [30]). The latter test had been applied in [30] to the problem of classification of even order $\mathbf{w}$–homogeneous integrable systems of the form (55). The results obtained can be summarised as follows:

The 4th order equations

It is easy to see that a homogeneous system (46) with $p = 4, r = 1$ is linear if $w_u > 3$. In the case $w_u = 3$ the only possibility is $F = \gamma u^2, \ \gamma \neq 0$ which leads to a non-integrable equation for any choice of α, β and γ. Thus the weight w_u can be equal to 2 or 1.

Case 1: The case of $w_u = 2$

The most general nonlinear homogeneous system of equations (46) with $p = 4, r = 1, w_u = 2$ (correspondingly $w_v = 3$) is of the form:

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_3 + \beta v_2 + c_1 u u_1 + c_2 uv, \end{cases} \tag{56}$$

where c_1, c_2 are arbitrary constants and at least one of them is not zero.

Without loss of generality we need to consider the following three types of system (56):

$$\begin{cases} u_t = v_1 \\ v_t = \frac{\mu^2 - \beta^2}{4} u_3 + \beta v_2 + c_1 u u_1 + c_2 uv, \ \mu \notin \{0, \pm\beta\} \end{cases} \tag{57}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 + c_1 u u_1 + c_2 uv \end{cases} \tag{58}$$

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{4} u_3 + v_2 + c_1 u u_1 + c_2 uv \end{cases} \tag{59}$$

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The first system represents the generic case $\alpha \notin \{0, -\frac{\beta^2}{4}\}$, while the other two represent the degenerate cases.

Theorem 13 *System (57) possesses two formal recursion operators with $\phi_{00}(\eta) = \eta$, $\psi_{00}(\eta) = 0$ and $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ if and only if $\beta = c_2 = 0$. By re-scalings it can be put in the form*

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 2uu_1 \end{cases} \tag{60}$$

Systems (58) and (59) are not integrable, they do not possess a formal recursion operator with $\phi_{00} = 0$, $\psi_{00} = \eta$ unless $c_1 = c_2 = 0$.

System (60) represents the Boussinesq equation (48), which is known to be integrable. In the Proposition 13 and below by re-scalings we mean an invertible change of variable of the form:

$$u \mapsto \alpha_1 u, \quad v \mapsto \alpha_2 v, \quad x \mapsto \alpha_3 x, \quad t \mapsto \alpha_4 t, \qquad \alpha_i \in \mathbb{C}.$$

Case 2: The case of $w_u = 1$

The most general homogeneous system of equations (46) with $p = 4, r = 1, w_u = 1$ is of the form

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_3 + \beta v_2 + c_1 uu_2 + c_2 u_1^2 + c_3 u_1 v + c_4 uv_1 + c_5 v^2 \\ \qquad + c_6 u^2 u_1 + c_7 u^2 v + c_8 u^4 \end{cases} \tag{61}$$

where c_i, $i = 1 \ldots 8$ are arbitrary constants and we assume that at least one of the coefficients $c_1, \ldots, c_5$ is not zero. Without loss of generality we consider the following three types of the system (61):

$$\begin{cases} u_t = v_1 \\ v_t = \frac{\mu^2 - \beta^2}{4} u_3 + \beta v_2 + c_1 uu_2 + c_2 u_1^2 + c_3 u_1 v + c_4 uv_1 + c_5 v^2 \\ \qquad + c_6 u^2 u_1 + c_7 u^2 v + c_8 u^4, \end{cases} \tag{62}$$

where $\mu \notin \{0, \pm\beta\}$, $\mu, \beta \in \mathbb{C}$ and

$$\begin{cases} u_t = v_1 \\ v_t = v_2 + c_1 uu_2 + c_2 u_1^2 + c_3 u_1 v + c_4 uv_1 + c_5 v^2 \\ \qquad + c_6 u^2 u_1 + c_7 u^2 v + c_8 u^4 \end{cases} \tag{63}$$

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{4} u_3 + v_2 + c_1 uu_2 + c_2 u_1^2 + c_3 u_1 v + c_4 uv_1 \\ \qquad + c_5 v^2 + c_6 u^2 u_1 + c_7 u^2 v + c_8 u^4 \end{cases} \tag{64}$$

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Theorem 14 *System (62) possesses two formal recursion operators with $\phi_{00}(\eta) = \eta$, $\psi_{00}(\eta) = 0$ and $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ if and only if (up to re-scalings) it is one of the list*

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + u_1^2 \end{cases} \tag{65}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 2u_1 v + 2u^2 u_1 \end{cases} \tag{66}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 2u_1 v + 4uv_1 - 6u^2 u_1 \end{cases} \tag{67}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_3 + 4uu_2 + 3u_1^2 - v^2 + 6u^2 u_1 + u^4 \end{cases} \tag{68}$$

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_3 + v_2 + 4\alpha uu_2 + 3\alpha u_1^2 + u_1 v + 2uv_1 - v^2 \\ \qquad +6\alpha u^2 u_1 + u^2 v + \alpha u^4, \ \alpha \neq -\frac{1}{4} \end{cases} \tag{69}$$

System (63) possesses a formal recursion operator with $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ if and only if (up to re-scalings) it is one of the list

$$\begin{cases} u_t = v_1 \\ v_t = v_2 + 2uv_1 \end{cases} \tag{70}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 - u_1^2 + 2u_1 v - v^2 \end{cases} \tag{71}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_2 - 2uu_2 - 2u_1^2 + 2u_1 v + 6uv_1 - 12u^2 u_1 \end{cases} \tag{72}$$

System (64) possesses two formal recursion operators with $\phi_{00}(\eta) = \eta$, $\psi_{00}(\eta) = 0$ and $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ if and only if (up to re-scalings) it is

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{4}u_3 + v_2 - uu_2 - \frac{3}{4}u_1^2 + u_1 v + 2uv_1 - v^2 \\ \qquad -\frac{3}{2}u^2 u_1 + u^2 v - \frac{1}{4}u^4. \end{cases} \tag{73}$$

Equation (65) is a potential version of the Boussinesq equation. Equations (66) and (67) are known to be integrable. Corresponding Lax representations and references can be found in [4].

Equations (68) and (69) can be mapped into linear equations

$$w_{tt} = w_4, \qquad w_{tt} = \alpha w_4 + w_{2t}$$

respectively by the Cole-Hopf transformation $u = (\log w)_x$.

Equation (70) can be reduced to the Burgers equation with time independent forcing

$$u_t = u_2 + 2uu_1 + w_1, \qquad w_t = 0$$

by invertible transformation $v = w + u_1 + u^2$. The latter can be linearised by the Cole-Hopf transformation.

Equation (71) can be reduced to the system

$$u_t = u_2 + w_1, \qquad w_t = -w^2$$

by a simple invertible the change of the variable $v = w + u_1$. System (71) provides an example of an equation that possesses neither higher symmetries nor a recursion operator. However, a formal recursion operator does exist and therefore it is in the list of the Proposition. Its integration can be reduced to the integration of a linear nonhomogeneous heat equation with a source term of a special form.

By a simple shift of the variable $v = w + u_1 + 2u^2$, system (72) can be transformed in the form

$$u_t = u_2 + 4uu_1 + w_1, \qquad w_t = 2D_x(uw). \tag{74}$$

System (72) possess an infinite hierarchy of symmetries of all orders generated by a recursion operator

$$\Re = -2u + 2u_1 D_x^{-1} + D_t D_x^{-1}$$

starting from the seed u_1.

Equation (73) is a particular case of (69) corresponding to the exceptional case $\alpha = -\frac{1}{4}$.

The 6th order equations

Homogeneous 6th order ($p = 6$) equations (46) with non-zero quadratic terms correspond to $w_u \leq 5$. We restrict ourself with the case $w_u > 0$. The weights $3, 4, 5$ do not lead to integrable equations:

Proposition 5 *For weights $w_u = 3, 4, 5$ there are no equations possessing a formal recursion operator.*

Case 1: The case of $w_u = 2$

The most general homogeneous system (46) with $p = 6, r = 1, w_u = 2$ can be written as

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_5 + \beta v_3 + D_x(c_1 uu_2 + c_2 u_1^2 + c_5 u^3) \\ \qquad + c_3 uv_1 + c_4 vu_1, \end{cases} \tag{75}$$

where $\alpha, \beta, c_i, i = 1, \ldots, 4$ are arbitrary constants and we assume that at least one of $c_1, \ldots, c_4$ is not zero.

Without loss of generality we consider the following three types of the above system:

$$\begin{cases} u_t = v_1 \\ v_t = \frac{\mu^2 - \beta^2}{4} u_5 + \beta v_3 + D_x(c_1 u u_2 + c_2 u_1^2 + c_5 u^3) \\ \quad + c_3 u v_1 + c_4 v u_1, \quad \mu \notin \{0, \pm\beta\}, \ \mu, \beta \in \mathbb{C} \end{cases} \tag{76}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + D_x(c_1 u u_2 + c_2 u_1^2 + c_5 u^3) + c_3 u v_1 + c_4 v u_1 \end{cases} \tag{77}$$

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{4} u_5 + v_3 + D_x(c_1 u u_2 + c_2 u_1^2 + c_5 u^3) \\ \quad + c_3 u v_1 + c_4 v u_1 \end{cases} \tag{78}$$

Theorem 15 *If system (76) possesses two formal recursion operators with $\phi_{00}(\eta) = \eta$, $\psi_{00}(\eta) = 0$ and $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ then, up to re-scalings, it is one of the list*

$$\begin{cases} u_t = v_1 \\ v_t = 2u_5 + v_3 + D_x(2uu_2 + u_1^2 + \frac{4}{27} u^3) \end{cases} \tag{79}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5} u_5 + v_3 + D_x(uu_2 + uv + \frac{1}{3} u^3) \end{cases} \tag{80}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5} u_5 + v_3 + D_x(2uu_2 + \frac{3}{2} u_1^2 + 2uv + \frac{4}{3} u^3) \end{cases} \tag{81}$$

If system (77) possesses a formal recursion operator with $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$ then, up to re-scalings, it is one of the list

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + uv_1 + u_1 v \end{cases} \tag{82}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + 2uu_3 + 4u_1 u_2 - 4u_1 v - 8uv_1 - 24u^2 u_1 \end{cases} \tag{83}$$

System (78) does not possess a formal recursion operator for any non-trivial choice of c_i.

Recursion operators and bi-Hamiltonian structure for these equations can be found in [30]. Lax representations can be found in [56, 57, 58, 30, 59].

Case 2: The case of $w_u = 1$

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Homogeneous systems of equations (46) with $p = 6, r = 1, w_u = 1$ can be written in the form:

$$
\begin{cases}
u_t = v_1 \\
v_t = \alpha u_5 + \beta v_3 + c_1 u_2^2 + c_2 u_1 u_3 + c_3 u u_4 + c_4 u_2 v + c_5 u_1 v_1 \\
\quad + c_6 u v_2 + c_7 v^2 + c_8 u_1^3 + c_9 u u_1 u_2 + c_{10} u^2 u_3 + c_{11} u^2 v_1 + \\
\quad + c_{12} u u_1 v + c_{13} u^2 u_1^2 + c_{14} u^3 u_2 + c_{15} u^3 v + c_{16} u^4 u_1 + c_{17} u^6,
\end{cases}
\tag{84}
$$

where $\alpha, \beta \in \mathbb{C}$, all $c_i, i = 1, \ldots, 17$ are arbitrary constants and at least one of $c_1, \ldots c_7$ is not zero.

We need to consider the following cases of the system (84):

$$
\begin{cases}
u_t = v_1 \\
v_t = \frac{\mu^2 - \beta^2}{4} u_5 + \beta v_3 + c_1 u_2^2 + c_2 u_1 u_3 + c_3 u u_4 + c_4 u_2 v \\
\quad + c_5 u_1 v_1 + c_6 u v_2 + c_7 v^2 + c_8 u_1^3 + + c_9 u u_1 u_2 + c_{10} u^2 u_3 \\
\quad + c_{11} u^2 v_1 + c_{12} u u_1 v + c_{13} u^2 u_1^2 + c_{14} u^3 u_2 \\
\quad + c_{15} u^3 v + c_{16} u^4 u_1 + c_{17} u^6
\end{cases}
\tag{85}
$$

where $\mu \notin \{0, \pm\beta\}$, $\mu, \beta \in \mathbb{C}$, and

$$
\begin{cases}
u_t = v_1 \\
v_t = v_3 + c_1 u_2^2 + c_2 u_1 u_3 + c_3 u u_4 + c_4 u_2 v + c_5 u_1 v_1 \\
\quad + c_6 u v_2 + c_7 v^2 + c_8 u_1^3 + c_9 u u_1 u_2 + c_{10} u^2 u_3 + c_{11} u^2 v_1 \\
\quad + c_{12} u u_1 v + c_{13} u^2 u_1^2 + c_{14} u^3 u_2 + c_{15} u^3 v + c_{16} u^4 u_1 + c_{17} u^6
\end{cases}
\tag{86}
$$

$$
\begin{cases}
u_t = v_1 \\
v_t = -\frac{1}{4} u_5 + v_3 + c_1 u_2^2 + c_2 u_1 u_3 + c_3 u u_4 + c_4 u_2 v + c_5 u_1 v_1 \\
\quad + c_6 u v_2 + c_7 v^2 + c_8 u_1^3 + c_9 u u_1 u_2 + c_{10} u^2 u_3 + c_{11} u^2 v_1 \\
\quad + c_{12} u u_1 v + c_{13} u^2 u_1^2 + c_{14} u^3 u_2 + c_{15} u^3 v + c_{16} u^4 u_1 + c_{17} u^6
\end{cases}
\tag{87}
$$

Theorem 16 *If system (85) possesses two formal recursion operators with* $\phi_{00}(\eta) = \eta, \psi_{00}(\eta) = 0,$ *and* $\phi_{00}(\eta) = 0, \psi_{00}(\eta) = \eta,$ *up to re-scalings, it is*

one of the equations in the following list

$$\begin{cases} u_t = v_1 \\ v_t = 2u_5 + v_3 + u_2^2 + 2u_1u_3 + \frac{4}{27}u_1^3 \end{cases} \tag{88}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5}u_5 + v_3 + u_1u_3 + u_1v_1 + \frac{1}{3}u_1^3 \end{cases} \tag{89}$$

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{5}u_5 + v_3 + 2u_1u_3 + \frac{3}{2}u_2^2 + 2u_1v_1 + \frac{4}{3}u_1^3 \end{cases} \tag{90}$$

$$\begin{cases} u_t = v_1 \\ v_t = \alpha u_5 + v_3 + 10\alpha u_2^2 + 15\alpha u_1u_3 + 6\alpha uu_4 + vu_2 \\ \quad +3u_1v_1 + 3uv_2 - v^2 + 15\alpha u_1^3 + 15\alpha u^2 u_3 + \\ \quad +60\alpha uu_1u_2 + 3uu_1v + 3u^2 v_1 + 45\alpha u^2 u_1^2 \\ \quad +20\alpha u^3 u_2 + u^3 v + 15\alpha u^4 u_1 + \alpha u^6, \ \alpha \neq -\frac{1}{4} \end{cases} \tag{91}$$

$$\begin{cases} u_t = v_1 \\ v_t = u_5 + 6uu_4 + 15u_1u_3 + 10u_2^2 - v^2 + 15u^2 u_3 + \\ \quad + 15u_1^3 + 60uu_1u_2 + 45u^2 u_1^2 + 20u^3 u_2 + 15u^4 u_1 + u^6 \end{cases} \tag{92}$$

If system (86) possesses a formal recursion operator with $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$, up to re-scalings, it is one of the list

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + u_1v_1 \end{cases} \tag{93}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 + 3u_1v_1 + 3uv_2 + 3u^2 v_1 \end{cases} \tag{94}$$

$$\begin{cases} u_t = v_1 \\ v_t = v_3 - u_2^2 + 2u_2v - v^2 \end{cases} \tag{95}$$

If system (87) possesses two formal recursion operators with $\phi_{00}(\eta) = \eta$, $\psi_{00}(\eta) = 0$, and $\phi_{00}(\eta) = 0$, $\psi_{00}(\eta) = \eta$, up to re-scalings, then it is

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{4}u_5 + v_3 - \frac{5}{2}u_2^2 - \frac{15}{4}u_1u_3 - \frac{3}{2}uu_4 + vu_2 \\ \quad +3u_1v_1 + 3uv_2 - v^2 - \frac{15}{4}u_1^3 - \frac{15}{4}u^2 u_3 - 15uu_1u_2 + 3uu_1v \\ \quad +3u^2 v_1 - \frac{45}{4}u^2 u_1^2 - 5u^3 u_2 + u^3 v - \frac{15}{4}u^4 u_1 - \frac{1}{4}u^6, \end{cases} \tag{96}$$

Recursion operators, bi-Hamiltonian structure and Lax representations of equations (88), (89), (90) and (93) are discussed in details in [30].

Equations (91),(92),(94) and (96) can be linearised by a Cole-Hopf type transformation [30].

Equation (95) has similar property as system (71). It can also be reduced to a triangular system

$$u_t = u_3 + w_1, \qquad w_t = -w^2$$

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in variables u and $w = v - u_2$.

10th order equations

In this section we present three examples of 10th order integrable non-evolutionary equations.

Proposition 6 *The following systems possess infinite hierarchies of higher symmetries:*

$$\begin{cases} u_t = v_1 \\ v_t = \frac{9}{64}u_9 + v_5 + D_x\left(3uu_6 + 9u_1u_5 + \frac{65}{4}u_2u_4 + \frac{35}{4}u_3^2 + \right. \\ \quad +2u_1v_1 + 4uv_2 + 20u^2u_4 + 80uu_1u_3 + 60uu_2^2 + 88u_1^2u_2 \\ \quad \left. +\frac{256}{5}u^3u_2 + \frac{384}{5}u^2u_1^2 + \frac{1024}{125}u^5\right) \end{cases} \quad (97)$$

$$\begin{cases} u_t = v_1 \\ v_t = -\frac{1}{54}u_9 + v_5 + \frac{5}{6}u_7u_1 + \frac{5}{3}u_6u_2 + \frac{5}{2}u_5u_3 + \frac{25}{12}u_4^2 \\ \quad -5u_3v_1 - \frac{15}{2}u_2v_2 - 10u_1v_3 - \frac{45}{4}u_5u_1^2 - \frac{75}{2}u_1u_2u_4 \\ \quad -\frac{75}{4}u_3^2u_1 - \frac{75}{4}u_2^2u_3 + \frac{45}{2}u_1^2v_1 + \frac{225}{4}u_3u_1^3 + \frac{675}{8}u_2^2u_1^2 - \frac{405}{16}u_1^5 \end{cases} \quad (98)$$

$$\begin{cases} u_t = v_1 \\ v_t = v_5 + 2u_2u_5 + 6u_3u_4 - 6u_3v - 22u_2v_1 - 30u_1v_2 \\ \quad -20uv_3 + 96uu_1v + 96u^2v_1 + 120D_x(4u^3u_2 + 6u^2u_1^2) \\ \quad -2D_x(8u^2u_4 + 32uu_1u_3 + 13u_1^2u_2 + 24uu_2^2) - 3840u^4u_1 \end{cases} \quad (99)$$

Bi-Hamiltonian structures and recursion operators for equations (97), (98) and (99) as well as the Lax representation for equation (97) can be found in [30]. System (99) has been considered independently by authors of [59, 60] and [61], where the corresponding Lax representation has been obtained. The Lax representations for equation (98) is still not known.

5.2 Odd order equations

In this section we formulate the diagonalisation method and globally classify homogeneous (with a positive weight $w_u > 0$) odd order ($p = 2n + 1$) equations (46) in the same spirit as we did for the scalar homogeneous evolution equations in Section 3. For the details, we refer the reader to the recent paper [31].

A symmetry of an odd order homogeneous equation (46) always starts with linear terms. A homogeneous symmetry in the symbolic representation starts either with $u\xi_1^m$ or with $v\zeta_1^{m+r}$. Without loss of generality we have

$$u_\tau = G = u\xi_1^m + \sum_{s \geq 2}\sum_{j=0}^{s} u^j v^{s-j} A_{j,s-j}(\xi_1, .., \xi_j, \zeta_1, .., \zeta_{s-j}), \quad m > 1 \quad (100)$$

or

$$u_\tau = G = v\zeta_1^{m+r} + \sum_{s \geq 2} \sum_{j=0}^{s} u^j v^{s-j} A_{j,s-j}(\xi_1, .., \xi_j, \zeta_1, .., \zeta_{s-j}), \quad m > 0 \quad (101)$$

Functions $A_{i,j}(\xi_1, \ldots, \xi_i, \zeta_1, \ldots, \zeta_j)$ in (100) and (101) are homogeneous polynomials in their variables, symmetric with respect to arguments $\xi_1, \ldots, \xi_i$ and $\zeta_1, \ldots, \zeta_j$. These functions can be explicitly determined in the terms of system (54), the symbolic representation of system (46), from the compatibility conditions.

Let us first concentrate on how to compute the Lie bracket between the linear part of system (54) denoted by K^1, i.e.

$$K^1 = \begin{pmatrix} v\zeta_1^r \\ u\xi_1^{2n+1-r} \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \eta^r \\ \eta^{2n+1-r} & 0 \end{pmatrix}$$

and any pair of differential polynomials. We know its symbolic representation takes simple and elegant form if matrix L is diagonal [23]. Inspired by this, we shall diagonalise matrix L, produce the required formula in new variables and then transform back to the original variables.

Notice that matrix L has two eigenvalues $\pm\eta^{n+\frac{1}{2}}$. Therefore, there exists a linear transformation

$$T = \begin{pmatrix} 1 & 1 \\ \eta^{n+\frac{1}{2}-r} & -\eta^{n+\frac{1}{2}-r} \end{pmatrix}$$

such that

$$T^{-1}LT = \text{diag}(\eta^{n+\frac{1}{2}}, -\eta^{n+\frac{1}{2}}).$$

Let us introduce new variables $\hat{u}$ and $\hat{v}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \hat{u} + \hat{v} \\ \hat{u}\xi_1^{n+\frac{1}{2}-r} - \hat{v}\zeta_1^{n+\frac{1}{2}-r} \end{pmatrix}.$$

Equally, we have

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = T^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u + \eta^{-n-\frac{1}{2}+r}(v) \\ u - \eta^{-n-\frac{1}{2}+r}(v) \end{pmatrix}.$$

The new variables $\hat{u}$ and $\hat{v}$ have the same weight, i.e., $w_{\hat{u}} = w_{\hat{v}} = w_u$. Without causing a confusion we assign the same symbols ξ and ζ for the symbolic representation of the ring generated by $\hat{u}$, $\hat{v}$ and their derivatives. The exponents of symbols can be half-integer, which corresponds to half-differentiation in x-space.

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Proposition 7 *In variables $\hat{u}$ and $\hat{v}$, system (54) takes the form*

$$\begin{cases} \hat{u}_t = \hat{u}\xi_1^{n+\frac{1}{2}} + \frac{1}{2g}\sum_{s\geq 2}\sum_{l=0}^{s}\hat{u}^l\hat{v}^{s-l}\hat{a}_{l,s-l}(\xi_1,..,\xi_l,\zeta_1,..,\zeta_{s-l}) \\ \hat{v}_t = -\hat{v}\zeta_1^{n+\frac{1}{2}} - \frac{1}{2g}\sum_{s\geq 2}\sum_{l=0}^{s}\hat{u}^l\hat{v}^{s-l}\hat{a}_{l,s-l}(\xi_1,..,\xi_l,\zeta_1,..,\zeta_{s-l}) \end{cases},$$

where $g = (\xi_1 + \cdots + \xi_l + \zeta_1 + \cdots + \zeta_{s-l})^{n+\frac{1}{2}-r}$ and $\hat{a}_{l,s-l}$ are defined in terms of $a_{i,s-i}(\xi_1,\ldots,\xi_i,\zeta_1,\ldots,\zeta_{s-i})$, $i = 0,\ldots,s$ as follows:

$$\hat{a}_{l,s-l}(\xi_1,..,\xi_l,\zeta_1,..,\zeta_{s-l}) = \sum_{i=0}^{s}\sum_{p=max\{0,l-s+i\}}^{min\{i,l\}} C_i^p C_{s-i}^{l-p}$$

$$(-1)^{s-i-l+p}\langle\langle a_{i,s-i}(\xi_1,..,\xi_p,\zeta_1,..,\zeta_{i-p},\xi_{p+1},\ldots,\xi_l,\zeta_{i-p+1},..,\zeta_{s-l})$$

$$(\xi_{p+1}\cdots\xi_j\zeta_{i-p+1}\cdots\zeta_{s-l})^{n+\frac{1}{2}-r}\rangle_{\mathcal{S}_l^\xi}\rangle_{\mathcal{S}_{s-l}^\zeta},$$

where C_i^j are binomial coefficients defined by $C_i^j = \frac{i!}{j!(i-j)!}$.

In variables $\hat{u}$ and $\hat{v}$, the linear parts of symmetries (100) and (101) of system (46) are also diagonal matrices. We can now derive the symmetry conditions for the transformed forms as in Theorem 2 for the scalar case. This leads to the explicit recursive relations between symmetry (100) or (101) and system (46).

With explicit formulas at hand, we can prove the following theorem, crucial for global classification:

Theorem 17 *Assume homogeneous system (46) ($p = 2n+1$) with $w_u > 0$ possesses a symmetry. Suppose there is another system of the same weight and of the same form*

$$\begin{cases} u_t = v\zeta_1^r, \\ v_t = u\xi_1^{2n+1-r} + \sum_{k\geq 2}\sum_{i=0}^{k} u^i v^{k-i} b_{i,k-i}(\xi_1,..,\xi_k,\zeta_1,..,\zeta_{k-i}), \end{cases} \quad (102)$$

whose quadratic terms equal to those of (46), that is, $b_{i,2-i}(x,y) = a_{i,2-i}(x,y)$, $i = 0,1,2$. Then if system (102) possesses a symmetry of the same order, then equation (102) and (46) are equal and sharing the same symmetry.

This can be viewed as another version of Theorem 4 for the systems case: the existence of infinitely many approximate symmetries of low degree together with one symmetry implies integrability.

Now we formulate the classification theorem:

Theorem 18 *If a homogeneous system (46) with a positive weight $w_u > 0$ of odd order ($p = 2n + 1$) possesses a hierarchy of infinitely many higher symmetries, then it is one of the systems in the following list up to re-scaling $u \mapsto \alpha u, v \mapsto \beta v, t \mapsto \gamma t, x \mapsto \delta x$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$:*

$$\begin{cases} u_t = v_1, \\ v_t = u_2 + 3uv_1 + vu_1 - 3u^2 u_1, \end{cases}$$

$$\begin{cases} u_t = v_1, \\ v_t = (D_x + u)^{2n}(u) - v^2, \quad n = 1, 2, 3, \ldots. \end{cases}$$

These systems can be rewritten in the form of non-evolutionary equations if we introduce a new variable $u = w_x$:

$$w_{tt} = w_{xxx} + 3w_x w_{t,x} + w_{xx} w_t - 3w_x^2 w_{xx}.$$

$$w_{tt} = (\partial_x + w_x)^{2n}(w_x) - w_t^2$$

The first equation is known to be integrable [53]. The second equation can be brought into linear $f_{tt} = \partial_x^{2n+1} f$ by the Cole-Hopf transformation $w = \log(f)$. Its symmetries are given by

$$u_{\tau_m} = D_x(D_x + u)^{m-1} u, \quad m = 2, 3, \ldots$$

and

$$u_{\tau_m} = D_x(v + D_t)(D_x + u)^{m-1} u.$$

These symmetries correspond to the symmetries $f_{\tau_m} = f_m$ and $f_{\tau_m} = f_{t,m}$ of equation $f_{tt} = f_{2n+1}$.

6 Symmetry structure of $(2+1)$–dimensional integrable equations

This section is devoted to the study of $(2 + 1)$–dimensional integrable equations. A famous example is the Kadomtsev-Petviashvili (KP) equation

$$u_t = u_{xxx} + 6uu_x + 3D_x^{-1} u_{yy}.$$

One of the main obstacles to extend the spectacular classification results of $(1 + 1)$-dimensional integrable equations to the $(2 + 1)$-dimensional case is that the equations themselves, their higher symmetries and conservation laws are non-local, i.e. they contain integral operators D_x^{-1} or D_y^{-1}. In 1998, Mikhailov and Yamilov, [45], introduced the concept of quasi-local functions

based on the observation that the operators D_x^{-1} and D_y^{-1} never appear alone but always in pairs like $D_x^{-1}D_y$ and $D_y^{-1}D_x$ for all known integrable equations and their hierarchies of symmetries and conservation laws, what enabled them to extend the symmetry approach of testing integrability [12] to the $(2+1)$-dimensional case.

The results in this section are based on a recent paper [35]. We develop the symbolic representation method to derive the hierarchies of $(2+1)$-dimensional integrable equations from the scalar Lax operators and to study their properties globally. We prove that these hierarchies are indeed quasi-local as conjectured by Mikhailov and Yamilov in 1998, [45].

6.1 Quasi-local polynomials and Symbolic representation

The basic definitions and notations of the ring of (commutative) differential polynomials are similar to in section 2.1. The derivatives of dependent variable u with respect to its independent variables x and y are denoted by $u_{ij} = \partial_x^i \partial_y^j u$. For small values of i and j, we sometimes write the indices out explicitly, that is u_{xxy} and u instead of u_{21} and u_{00}.

A differential monomial takes the form $u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n}$. We call n the degree of the monomial. Let $\mathcal{R}^n$ denote the set of differential polynomials of degree n. The ring of differential polynomials is denoted by $\mathcal{R} = \oplus_{n \geq 1} \mathcal{R}^n$. It is a differential ring with total x-derivation and y-derivation

$$D_x = \sum_{i,j \geq 0} u_{i+1,j} \frac{\partial}{\partial u_{ij}} \qquad \text{and} \quad D_y = \sum_{i,j \geq 0} u_{i,j+1} \frac{\partial}{\partial u_{ij}}.$$

Let us denote

$$\Theta = D_x^{-1} D_y, \qquad \Theta^{-1} = D_y^{-1} D_x.$$

The concept of **quasi-local (commutative) polynomials** $\mathcal{R}_\Theta$ was introduced in [45]. Its definition is similar to the ring extension of $\mathcal{R}_H$ as in section 4.1. We consider a sequence of extensions of $\mathcal{R}$ as follows:

$$\mathcal{R}_{\Theta^0} = \mathcal{R}, \quad \mathcal{R}_{\Theta^{n+1}} = \overline{\mathcal{R}_{\Theta^n} \bigcup \Theta(\mathcal{R}_{\Theta^n}) \bigcup \Theta^{-1}(\mathcal{R}_{\Theta^n})},$$

where the sets

$$\Theta(\mathcal{R}_{\Theta^n}) = \{\Theta(f); f \in \mathcal{R}_{\Theta^n}\}, \quad \Theta^{-1}(\mathcal{R}_{\Theta^n}) = \{\Theta^{-1}(f); f \in \mathcal{R}_{\Theta^n}\}$$

and the horizontal line denotes the ring closure. Each $\mathcal{R}_{\Theta^n}$ is a ring and the index n indicates the nesting depth of the operators $\Theta^{\pm 1}$. Clearly, we have

$$\mathcal{R}_{\Theta^0} \subset \mathcal{R}_{\Theta^1} \subset \mathcal{R}_{\Theta^2} \subset \cdots \subset \mathcal{R}_{\Theta^n} \subset \cdots \subset \mathcal{R}_{\Theta^\infty} = \mathcal{R}_\Theta.$$

To define the symbolic representation of $\mathcal{R}$, we replace u_{ij} by $\hat{u}\xi^i\eta^j$, where ξ and η are symbols (comparing to the definitions in section 2.3).

Definition 10 *The symbolic representation of a differential monomial is defined as*

$$u_{i_1,j_1} u_{i_2,j_2} \cdots u_{i_n,j_n} \longmapsto \frac{\hat{u}^n}{n!} \sum_{\sigma \in \mathcal{S}_n} \xi_{\sigma(1)}^{i_1} \eta_{\sigma(1)}^{j_1} \cdots \xi_{\sigma(n)}^{i_n} \eta_{\sigma(n)}^{j_n} \ .$$

The result of action of operators $\Theta^{\pm 1}$ on a monomial

$$\hat{u}^k a(\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k)$$

in the symbolic representation is given by

$$\hat{u}^k a(\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k) \left(\frac{\eta_1 + \cdots + \eta_k}{\xi_1 + \cdots + \xi_k} \right)^{\pm 1} .$$

By induction, we can define the symbolic representation of any element in $\mathcal{R}_\Theta$, which is a rational function with its denominator being the products of the linear factors. The expression of the denominator uniquely determines how the operator $\Theta^{\pm 1}$ is nested. For example, the symbol of $u_x(\Theta u)\Theta^{-1}u \in \mathcal{R}_{\Theta^1}$ is

$$\frac{\hat{u}^3}{3}(\xi_1 \frac{\eta_2}{\xi_2} \frac{\xi_3}{\eta_3} + \xi_2 \frac{\eta_1}{\xi_1} \frac{\xi_3}{\eta_3} + \xi_3 \frac{\eta_2}{\xi_2} \frac{\xi_1}{\eta_1}) \ .$$

The symbolic representations of pseudo-differential operators in $(2+1)$–dimensional case are similar to for the case of one spatial variable in section [35]. However, we assign a special symbol X to the operator D_x in contrast to the symbol Y for the operator D_y.

The extension of the symbolic representation from one dependent variable to several dependent variables is straightforward. We need to assign new symbols for each of them such as assigning $\xi^{(1)}, \eta^{(1)}$ for u and $\xi^{(2)}, \eta^{(2)}$ for v and so on.

6.2 Lax formulation of $(2+1)$–dimensional integrable equations

We give a short description of construction of $(2+1)$-dimensional integrable equations from a given scalar Lax operator based on the well-known Sato approach. For details on the Sato approach for the $(1+1)$-dimensional case, see the recent books [62, 63] and related references in them.

Let H be an m-th order pseudo-differential operator in two spatial variables of the form

$$H = -D_y + a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots , \quad m \geq 0,$$

where coefficients a_k are functions of x, y. Let the commutator be the bracket on the set of pseudo-differential operators. Thus, the set of pseudo-differential operators forms a Lie algebra. For an integer $k < m$, we split into

$$H_{\geq k} = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_k D_x^k$$
$$H_{<k} = H - H_{\geq k} = -D_y + a_{k-1} D_x^{k-1} + \cdots$$

This operator algebra decomposes as a direct sum of two subalgebras in both commutative and noncommutative cases when $k \in \{0, 1\}$. Similar to the $(1+1)$-dimensional case, such decompositions are naturally related with integrability and lead to admissible scalar Lax operators for the case of $(2+1)$ dimensions:

a. $k = 0:$ $n \geq 2,$
$$L = D_x^n + u^{(n-2)} D_x^{n-2} + u^{(n-3)} D_x^{n-3} + \cdots + u^{(0)} - D_y;$$

b. $k = 1:$ $n \geq 2,$
$$L = D_x^n + u^{(n-1)} D_x^{n-1} + u^{(n-2)} D_x^{n-2} + \cdots + u^{(0)} + D_x^{-1} u^{(-1)} - D_y;$$

c. $k = 1:$ $L = u^{(0)} + D_x^{-1} u^{(-1)} - D_y;$

where $u^{(i)}$ are functions of two spatial variables x, y. We often use $u, v, w, \cdots$ in the examples. For the KP equation, the Lax operator is the case **a** when $n = 2$, namely, $L = D_x^2 + u - D_y$.

Let $S = D_x + a_0 + a_{-1} D_x^{-1} + \cdots$. For any operator L listed in cases **a**, **b** and **c**, the relation

$$[S, \ L] := SL - LS = 0, \tag{103}$$

uniquely determines the operator S by taking the integration constants to be zeros. Furthermore, we have $[S^n, \ L] = 0$ for any $n \in \mathbb{N}$. For each choice of i, we introduce a different time variable t_i and define the Lax equation by

$$\frac{\partial L}{\partial t_i} = [S^i{}_{\geq k}, L], \tag{104}$$

where k is determined by the operator L as listed in cases **a**, **b** and **c**.

Theorem 19 *For the operator S uniquely determined by (103), the flows defined by Lax equations (104) commute, i.e., $\partial_{t_j}\partial_{t_i}L = \partial_{t_i}\partial_{t_j}L$.*

6.3 Lax formulation in symbolic representation

We put the formalism of section 6.2 into the symbolic form. The strategy is to do the calculation as much as possible without symmetrisation and only perform the symmetrisation at the last stage to get the uniqueness of the symbolic representation since the symmetrisation complicates the calculation dramatically.

Let us assign the symbols $\xi^{(i)}, \eta^{(i)}$ for dependent variable $u^{(i)}$. The symbolic representations of the admissible scalar Lax operators are

a. $k = 0:$ $n \geq 2,$
   $L = X^n - Y + \hat{u}^{(n-2)}X^{n-2} + \hat{u}^{(n-3)}X^{n-3} + \cdots + \hat{u}^{(0)}$;

b. $k = 1:$ $n \geq 2,$
   $\hat{L} = X^n - Y + \hat{u}^{(n-1)}X^{n-1} + \cdots + \hat{u}^{(0)} + \hat{u}^{(-1)}\frac{1}{X+\xi^{(-1)}}$;

c. $k = 1:$ $\hat{L} = -Y + \hat{u}^{(0)} + \hat{u}^{(-1)}\frac{1}{X+\xi^{(-1)}}$;

Here we only treat the case **a**. The study of the cases **b** and **c** can be found in [35].

It is convenient to consider formal series in the form

$$S = X + \sum_{i=0}^{n-2} \hat{u}^{(i)} a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) \tag{105}$$

$$+ \ \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{n-2} \hat{u}^{(i_1)}\hat{u}^{(i_2)} a_2^{(i_1 i_2)}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X) + \cdots,$$

where $n \geq 2$ and a_i are functions of their specific arguments, the superindex $i_s \in \{0, 1, 2, \cdots, n-2\}$ and the subindex j_k is defined by the number of i_k

in the list of $[i_1, i_2, \cdots, i_l]$. This implies that $j_1 = 1$ and $j_k \geq 1$, $k = 1, 2, \cdots$. For example, when $i_1 = i_2$, the arguments of function $a_2^{i_1 i_1}$ are $\xi_1^{(i_1)}$, $\eta_1^{(i_1)}$, $\xi_2^{(i_1)}$, $\eta_2^{(i_1)}$ and X.

It is easy to check that

$$[X^n - Y, \ \phi(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)] =$$
$$= \ N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)\phi,$$

where the polynomial N_l is defined by

$$N_l(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_l, \eta_l; X) = (\sum_{i=1}^{l} \xi_i + X)^n - X^n - \sum_{i=1}^{l} \eta_i \ .$$

Proposition 8 *For any operator L in case* **a**, *if the formal series (105) satisfies the relation $[S, \ L] = 0$ (cf. (103)), we have for $l \geq 1$,*

$$a_l^{(i_1 i_2 \cdots i_l)} = a_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)$$
$$= \ \prod_{r=1}^{l}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} b_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, .., \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X).$$

The function b_l, $l \geq 1$, is defined by

$$b_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) =$$
$$= \ \frac{c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}{N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}$$

with

$$c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) =$$
$$= \ b_{l-1}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)} \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})$$
$$- b_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X), \quad l > 1$$

and the initial function $c_1(\xi_1^{(i)}, \eta_1^{(i)}, X) = \xi_1^{(i)}$.

To construct the hierarchy of the Lax equations we need to expand the coefficients of operator (105) at $X \to \infty$ and truncate at the required degree. When $n \geq 2$, the expansion of $N_l(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_l, \eta_l; X)^{-1}$ at $X \to \infty$ is

of the form

$$N_l(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_l, \eta_l; X)^{-1}$$

$$= \frac{1}{nX^{n-1}(\sum_{i=1}^{l} \xi_i)} \sum_{j \geq 0} \left\{ \frac{\sum_{i=1}^{l} \eta_i}{nX^{n-1}(\sum_{i=1}^{l} \xi_i)} \right.$$

$$\left. - \frac{1}{n} \sum_{k=0}^{n-2} C_n^k \, X^{k+1-n} (\sum_{i=0}^{l} \xi_i)^{n-k-1} \right\}^{j}, \quad n \geq 2.$$

Therefore, if we want to prove that the coefficients of operators (105), i.e. the functions a_l, are quasi-local defined similarly as definition 7, we need to show that the functions c_l can be split into the sum of the image of D_x and the image of D_y. It is clear that $a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X)$ are quasi-local since we have $c_1(\xi_1^{(i)}, \eta_1^{(i)}, X) = \xi_1^{(i)}$. When $l > 1$, we have

Proposition 9 *The functions $c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)$ for $l > 1$ vanish after substitution*

$$\xi_{j_1}^{(i_1)} = -\xi_{j_2}^{(i_2)} - \cdots - \xi_{j_{l-1}}^{(i_{l-1})} \quad and \quad \eta_{j_1}^{(i_1)} = -\eta_{j_2}^{(i_2)} - \cdots - \eta_{j_{l-1}}^{(i_{l-1})}.$$

In fact, this proposition does not lead to our intended conclusion that b_l and thus a_l are quasi-local since the objects are rational, not polynomial. For example, the expression $u^2(\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1})$ representing $u\Theta u - (\Theta u)u$ satisfies the above proposition. However, we can not write $u\Theta u - (\Theta u)u = D_x f_1 + D_y f_2$, where both f_1 and f_2 are in $\mathcal{R}_\Theta$.

Notice that the formulae in Proposition 8 are without symmetrisation. Combining these expressions, we can obtain the formulae of high degree terms of operator S when dependant variables are commuting. Every term is such S is quasi-local. This implies that every term in S^n is quasi-local. From Theorem 19 follows

Theorem 20 *The hierarchies of commutative $(2 + 1)$-integrable equation with scalar Lax operators are quasi-local.*

The above setting up is valid for the noncommutative case except Theorem 20. However, the extension of the concept of quasi-locality to the noncommutative case is rather complicated. D_x and D_y are the only derivations for the commutative differential ring. The extension simply enables us to apply D_x^{-1} and D_y^{-1} on the derivations. We know that the commutators are also derivations for a noncommutative associative algebra and we need to take them into consideration. There are some further discussions on this topic in [35].

Summary and discussion

In this article we have reviewed some recent developments in the symmetry approach in the symbolic representation. In particular we have discussed two different methods. One method is based on the study of conditions for the existence of a formal recursion operator, another one is based on the explicit analysis of approximate symmetries of low degrees.

The formal recursion method is a very efficient tool for testing the integrability of a given PDE. This method is based only on the fact of existence of an infinite hierarchy of higher symmetries and is not sensitive to possible lacunas in the hierarchy of symmetries. It is extendable to wide classes of equations, including certain types of non-local equations. The method is rather simple and convenient for classification of integrable systems of a fixed order. We have illustrated its power in applications to classification of integrable generalisations of Boussinesq, Benjamin-Ono and Camassa-Holm type equations.

Explicit analysis of approximate symmetries relies on the structure of dispersion relations (linear terms) of systems. We have seen that the structure of symmetries of a given equation is parametrised by their dispersion relations and the analysis of existence of approximate symmetries is based on divisibility properties of special polynomials determined by the dispersion laws. Such divisibility properties are often obtained via algebraic geometry and number theoretic methods. It gives a detailed information on the structure of the hierarchy of higher symmetries. So far it is the only method which prove to be suitable for a global classification. Here we mean a classification of integrable equations in all orders. In the frame of this method it has been demonstrated that any integrable homogeneous evolutionary equation

$$u_t = u_n + f(u_{n-1}, \dots, u) \qquad n \geq 2,$$

where $w(u) \geq 0$ is a symmetry (a member of a hierarchy) of one of the equation of order $2, 3$ or 5 presented in Theorem 9.

Systems of equations is a considerably more complicated object. Here the only "global" result available is a classification of integrable homogeneous Boussinesq type equations of odd order (Section 5.2). In the theory of integrable systems the description of admissible structures of linear terms (the dispersion laws) for equations and their symmetries is important and yet unsolved problem. It is so called *spectrum* or *dispersion* problem. Let us consider system of PDEs

$$\mathbf{u}_t = A\mathbf{u}_n + \mathbf{F}(\mathbf{u}_{n-1}, \dots, \mathbf{u}), \quad \mathbf{u} = (u_1, \dots, u_N)^T, \quad n \geq 2 \qquad (106)$$

where A is a constant $N \times N$ matrix. Its higher symmetry, if it exists, is of the form
$$\mathbf{u}_{\tau_m} = B(m)\mathbf{u}_m + \mathbf{G}(\mathbf{u}_{m-1}, \ldots, \mathbf{u}), \qquad m \geq 2$$
where $B(m)$ is a constant matrix commuting with A. Let us denote $\lambda_1, \ldots, \lambda_N$ the eigenvalues of the matrix A and assume that $\lambda_1 \neq 0$, then the set $S_A = \{\lambda_2/\lambda_1, \ldots, \lambda_N/\lambda_1\}$ is called the *spectrum* of system (106). Similarly defined $S_{B(m)} = \{\mu_2(m)/\mu_1(m), \ldots, \mu_N(m)/\mu_1(m)\}$ the spectrum of symmetry, where $\mu_1(m), \ldots, \mu_N(m)$ are eigenvalues of the matrix $B(m)$. The spectrum is invariant with respect to any re-scaling. In many respects it reflects properties of symmetries, conservation laws and solutions of the system. For example, the existence of higher conservation laws for a system of two even order equations implies that $S_A = \{-1\}$ (see [20]).

Integrable system [57]
$$\begin{cases} u_t = (5 - 3\sqrt{5})u_3 - 2uu_1 + (3 - \sqrt{5})vu_1 + 2uv_1 + (1 + \sqrt{5})vv_1, \\ v_t = (5 + 3\sqrt{5})v_3 + (1 - \sqrt{5})uu_1 + 2vu_1 + (3 + \sqrt{5})uv_1 - 2vv_1, \end{cases}$$

has spectrum $S_A = \{\lambda_2/\lambda_1\} = \{-\frac{1}{2}(7 + 3\sqrt{5})\}$. It possesses an infinite dimensional algebra of symmetries of orders $m \equiv 1, 3, 7, 9 \mod 10$. The ratio of parameters $\mu_2(m)/\mu_1(m)$ is given by

$$\frac{\mu_2(m)}{\mu_1(m)} = \frac{\left(1 + \exp\left(\frac{2\pi i}{5}\right)\right)^m}{1 + \exp\left(\frac{2m\pi i}{5}\right)}.$$

Symmetry approach in symbolic representation can be applied to the study of possible spectra and classification of integrable systems. For systems of two homogeneous differential polynomial equations of second order the problem has been solved in [23]. Recently we have been working on this problem for systems of two equations of odd order. The results obtained will be published elsewhere. Here we present two rather non-trivial examples of integrable systems, which we believe are new.

The following system

$$\begin{cases} u_t &= (9 - 5\sqrt{3})u_5 + D_x\left\{2(9 - 5\sqrt{3})uu_2 + (-12 + 7\sqrt{3})u_1^2\right\} \\ &\quad + 2(3 - \sqrt{3})u_3 v + 2(6 - \sqrt{3})u_2 v_1 + 2(3 - 2\sqrt{3})u_1 v_2 \\ &\quad - 6(1 + \sqrt{3})uv_3 + D_x\left\{2(33 + 19\sqrt{3})vv_2 + (21 + 12\sqrt{3})v_1^2\right\} \\ &\quad + \frac{4}{5}(-12 + 7\sqrt{3})u^2 u_1 + \frac{8}{5}(3 - 2\sqrt{3})(vuu_1 + u^2 v_1) \\ &\quad + \frac{4}{5}(24 + 13\sqrt{3})v^2 u_1 + \frac{8}{5}(36 + 20\sqrt{3})uvv_1 - \frac{8}{5}(45 + 26\sqrt{3})v^2 v_1, \\[2mm] v_t &= (9 + 5\sqrt{3})v_5 + D_x\left\{2(33 - 19\sqrt{3})uu_2 + (21 - 12\sqrt{3})u_1^2\right\} \\ &\quad - 6(1 - \sqrt{3})u_3 v + 2(3 + 2\sqrt{3})u_2 v_1 + 2(6 + \sqrt{3})u_1 v_2 \\ &\quad + 2(3 + \sqrt{3})uv_3 + D_x\left\{2(9 + 5\sqrt{3})vv_2 - (12 + 7\sqrt{3})v_1^2\right\} \\ &\quad - \frac{8}{5}(45 - 26\sqrt{3})u^2 u_1 + \frac{8}{5}(36 - 20\sqrt{3})vuu_1 + \frac{4}{5}(24 - 13\sqrt{3})u^2 v_1 \\ &\quad + \frac{8}{5}(3 + 2\sqrt{3})(v^2 u_1 + uvv_1) - \frac{4}{5}(12 + 7\sqrt{3})v^2 v_1 \end{cases}$$

possesses an infinite dimensional algebra of higher symmetries with

$$\frac{\mu_2(m)}{\mu_1(m)} = \frac{(1 + \exp(\frac{\pi i}{6}))^m}{1 + \exp(\frac{m\pi i}{6})}, \qquad m \equiv 1, 5, 7, 11 \mod 12.$$

System

$$\begin{cases} u_t &= -\frac{5}{3}u_5 - 10vv_3 - 15v_1v_2 + 10uu_3 + 25u_1u_2 - 6v^2v_1 \\ &\quad +6v^2u_1 + 12uvv_1 - 12u^2u_1, \\ v_t &= 15v_5 + 30v_1v_2 - 30v_3u - 45v_2u_1 - 35v_1u_2 - 10vu_3 \\ &\quad -6v^2v_1 + 6v^2u_1 + 12u^2v_1 + 12vuu_1. \end{cases}$$

possesses symmetries of orders $m \equiv 1, 5 \mod 6$ with

$$\frac{\mu_2(m)}{\mu_1(m)} = \frac{(1 + \exp(\frac{\pi i}{3}))^m}{1 + \exp(\frac{m\pi i}{3})}.$$

There is a reduction $v = 0$ to the Kaup-Kupershmidt equation.

Acknowledgments

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